Linear Algebra, MATH–410 Gregor Kovačič

Homework #6 Due: Wednesday, October 13, 1999

1. Suppose that A is an $n \times n$ real symmetric matric of rank $n-1$. Suppose also that $Au = 0$, where $u = (1, 1, \dots, 1)^T$. Show that $Ax = y$ has a solution if and only if $\sum_{j=1}^{n} y_j = 0$.

2. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & -1 \\ 1 & 2 & 4 & 3 \end{pmatrix}.$$

(a) Find a basis for $\operatorname{Ker} A^*$.

(b) Find a basis for $(\operatorname{Ker} A^*)^\perp$.

(c) Find vectors b_i such that $y \perp b_i$ implies $Ax = y$ is solvable.

3. Show that the vectors $u_1 = (1/9, 4/9, 8/9)$, $u_2 = (8/9, -4/9, 1/9)$, and $u_3 = (-4/9, -7/9, 4/9)$ form an orthonormal basis of $\mathbb{R}^3$. Find the projection of any vector $x = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ onto the linear span of the vectors u_1 and u_2, and also the normal from any vector to that same linear span. Both, taking the projection and the normal, are linear operators. Write down their matrices in the usual basis of $\mathbb{R}^3$.

4. Show that any real $n \times n$ matrix can be written as a sum of a symmetric and an anti-symmetric matrix.

5. Let A be any $m \times n$ matrix. Show that the matrices A^*A and AA^* are both positive. (Recall that a matrix B is positive if $(Bx, x) \geq 0$.

6. Let U be a unitary transformation on $\mathbb{C}^n$, that is, $UU^* = U^*U = I$. Show that it preserves the scalar product, that is $(Ux, Uy) = (x, y)$. Deduce that U therefore preserves lengths of vectors.

7. Show that a symmetric real $n \times n$ matrix A equals zero if and only if $(Ax, x) = 0$ for all vectors x.

HINT: Expand $(A(x + y), x + y)$.

8. Show that a linear transformation C on $\mathbb{R}^n$ is orthogonal of and only if it preserves lengths of vectors.

HINT: Apply the result of Problem 7 to the transformation $C^*C - I$.

9. <u>Extra Credit:</u> Let A be an $m \times n$ matrix. Then

(a) $\operatorname{Ker} A = \operatorname{Ker} A^*A$ and $\operatorname{Ker} A^* = \operatorname{Ker} AA^*$,

(b) $\operatorname{Im} A = \operatorname{Im} AA^*$ and $\operatorname{Im} A^* = \operatorname{Im} A^*A$,

(c) $\operatorname{Rank} A = \operatorname{Rank} AA^* = \operatorname{Rank} A^*A$.