Linear Algebra, MATH–410 Gregor Kovačič

Homework #4 Due: Wednesday, September 29, 1999

1. Consider the linear transformation of $\mathbb{R}^3$ given by $A\mathbf{x} = (\mathbf{a}\cdot\mathbf{x})\,\mathbf{a} + |\mathbf{a}|^2\,\mathbf{x}$.

(a) What are the range (image) and kernel of A?

(b) Find the matrix of A in the usual basis $\mathbf{e}_j$.

2. Define the linear map $A : \mathcal{P}_2 \to \mathbb{R}^2$ by the formula

$$Ap = \left(\int_0^1 p(t)\,dt, \int_0^2 p(t)\,dt, \int_0^3 p(t)\,dt, \right) = (x_1, x_2, x_3).$$

(a) Compute the matrix of A in the basis 1, t, t^2 in $\mathcal{P}_2$ and the usual basis in $\mathbb{R}^3$.

(b) Can you recover $p(t)$ from (x_1, x_2, x_3), and if yes, how?

(c) Compute A^{-1} if it exists. Is this computation legitimate even though A maps one vector space into a different vector space?

3. (a) What is the effect of multiplying any vector $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$ by the matrix

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}?$$

What is A^{-1}? What is $\mathbf{x}\cdot A\mathbf{y}$?

(b) Consider $\mathbb{C}$ as a two-dimensional real vector space $\mathbb{R}^2$. Consider the linear map $z \mapsto e^{i\theta}z$ on $\mathbb{C}$. What is the matrix of this map on $\mathbb{R}^2$ in the usual basis?

4. Let the linear map $T : \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$$T(x, y, z) = (x + y + z, x + 2y - 3z, 2x + 3y - 2z, 3x + 4y - z).$$

Find bases for the image and the kernel of T.

HINT: First find the matrix for T in the usual bases.

5. Given an $n \times n$ matrix A, show that if there exist matrices B and C such that $AB = I$ and $CA = I$ where I is the identity matrix, then $B = C = A^{-1}$.

6. Let the linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the formula $S(x, y, z) = (x + 2y - 3z, 2x + y + z, 3y + 5z)$.

(a) Find the matrix of this transformation in the basis $u_1 = (1, 1, 0)$, $u_2 = (1, 2, 3)$, $u_3 = (1, 3, 5)$.

(b) Find the components of the vector Sv, where $v = (1, 1, 1)$ with respect to the basis u_1, u_2, u_3.

7. Find the change-of-basis matrix P from the usual basis in $\mathbb{R}^3$ to the basis $w_1 = (1, 1, 1)$, $w_2 = (1, 1, 0)$, and $w_3 = (1, 0, 0)$. Also, find the change-of-basis matrix Q from the basis w_1, w_2, w_3 back to the usual basis.

8. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be given by the matrix

$$A = \begin{pmatrix} 3 & -5 \\ 2 & 7 \end{pmatrix}$$

in the usual basis. What is the matrix of the operator A in the basis $v_1 = (1, 3)$, $v_2 = (2, 5)$.

9. The linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ has the property that $A(3, 1) = (2, -4)$ and $A(1, 1) = (0, 2)$. What is the matrix of A in the usual basis?