Linear Algebra, MATH–410 Gregor Kovačič

Homework #3 Due: Wednesday, September 22, 1999

1. Verify that the following expressions define inner products in the indicated spaces:

(a) $(x, y) = \sum_{j=1}^{n} \omega_j x_j \overline{y_j}$, $\omega_j > 0$, in $\mathbb{C}^n$.

(b) $(f, g) = \sum_{i=1}^{n} f(x_i)\overline{g(x_i)}$ in the space of discrete functions defined at the points $x_1 < x_2 < \cdots < x_n$.

(c) $(f, g) = \int_a^b f'(x)\overline{g'(x)}w(x)\,dx + f(a)\overline{g(a)}$, $w(x) > 0$, in the space of continuously differentiable functions on the interval $[a, b]$.

2. (a) Show that the functions 1, $\cos nx$, $\sin nx$, $n = 1, 2, 3, \ldots$, form an orthogonal set for the inner product

$$(f, g) = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx \tag{1}$$

on the space of continuous functions. What is the corresponding orthonormal set.

(b) Let

$$f(x) = \frac{c_0}{2} + \sum_{k=1}^{n} \left(c_k \cos kx + d_k \sin kx\right).$$

Use the inner product (1) to compute the coefficients c_k and d_k.

3. Extra credit: Define the *Chebyshev Polynomials* by the formula

$$T_n(x) = \cos(n\theta), \quad x = \cos(\theta).$$

(a) Use trigonometric identities to show that

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1.$$

(b) Compute $T_n(x)$ for $n = 0, \ldots 4$.

(c) Show that T_n is a polynomial of degree n with leading coefficient 2^{n-1} if $n > 0$.

(d) Show that T_n is an odd function for n odd and an even function for n even.

(e) Show that the sequence of T_k's is orthogonal with respect to the inner product

$$(f, g) = \int_{-1}^{1} \frac{f(x)\overline{g(x)}}{\sqrt{1 - x^2}}dx.$$

(f) Find $\|T_n\|^2$ for $n = 0, 1, 2, \ldots$.

4. On $\mathbb{R}^3$, define

$$A(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{a})\,\mathbf{a} + 2(\mathbf{x} \cdot \mathbf{b})\,\mathbf{b},$$

where $\cdot$ denotes the usual inner product. Here

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Show that A is linear, and find its matrix in the usual basis

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

5. The transformation A is defined on the space $\mathcal{P}_2$ of polynomials of degree ≤ 2 by $Ap(t) = p''(t) + 2p'(t) + p(t)$. Find the matrix of this transformation in the basis $\{1, t, t^2\}$. What is $\operatorname{Ker} A$?

6. Compute AB and BA for the matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 3 \\ -1 & 4 & 1 \end{pmatrix}$$

if the products exist.

7. Let A and B be $n \times n$ real matrices. Show that if $(x, Ay) = (x, By)$ for all x and y in $\mathbb{R}^n$, then $A = B$.

8. Suppose that $Ap(t) = p(t+1)$ for every polynomial p in $\mathcal{P}_n$. Prove that if D is the differentiation operator ($Dp(t) = p'(t)$), then

$$A = 1 + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^{n-1}}{(n-1)!}.$$

9. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Compute AB and BA? Are they equal?