

1. If the vectors x , y , z , and u in the space \mathcal{P} of polynomials are defined by $x(t) = 1$, $y(t) = t$, $z(t) = t^2$, and $u(t) = 1 + t + t^2$, show that x , y , z , and u are linearly dependent, but any three of them are linearly independent.

2. (a) Find two bases in \mathbb{R}^4 such that the only vectors in common to both are $(0, 0, 1, 1)$ and $(1, 1, 0, 0)$.

(b) Find two bases in \mathbb{R}^4 that have no vectors in common so that one of them contains the vectors $(1, 0, 0, 0)$ and $(1, 1, 0, 0)$ and the other contains the vectors $(1, 1, 1, 0)$ and $(1, 1, 1, 1)$.

3. If \mathcal{X} is the set consisting of the six vectors $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, $(1, 0, 0, 1)$, $(0, 1, 1, 0)$, $(0, 1, 0, 1)$, and $(0, 0, 1, 1)$ in \mathbb{R}^4 , find two different bases that are subsets of \mathcal{X} .

4. In the space of polynomials \mathcal{P}_2 find an orthonormal basis, starting from the vectors 1 , t , and t^2 . The scalar product is $(p, q) = \int_{-1}^1 p(t)q(t) dt$.

5. (a) Two vectors x and y in \mathbb{R}^n are orthogonal iff $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

(b) Show that (a) is false in \mathbb{C}^n .

(c) Two vectors x and y in \mathbb{R}^n are orthogonal iff $\|\alpha x + \beta y\|^2 = \|\alpha x\|^2 + \|\beta y\|^2$ for all pairs of scalars α and β .

(d) If x and y are vectors in \mathbb{R}^n , and if $\|x\| = \|y\|$, then $x - y$ and $x + y$ are orthogonal. (Picture?) Discuss the corresponding statement for \mathbb{C}^n .

(e) If x and y are vectors in an inner product space, then $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$. Picture?

6. (a) Let e_1, e_2, \dots, e_n be an orthonormal basis of an inner-product space. Show Bessel's inequality

$$\sum_{j=1}^k |(x, e_j)|^2 \leq \|x\|^2$$

for all $1 \leq k \leq n$.

(b) Deduce from (a) the Cauchy-Schwartz inequality $|(x, y)| \leq \|x\| \|y\|$.

7. Let

$$U = \text{span}\{(1, 1, 0, -1), (1, 2, 3, 0), (2, 3, 3, -1)\},$$
$$W = \text{span}\{(1, 2, 2, -2), (2, 3, 2, -3), (1, 3, 4, -3)\}.$$

Find bases and dimensions for U , W , $U + W$, and $U \cap W$.

8. Suppose U and W are distinct 4-dimensional subspaces of \mathbb{R}^6 . Find all possible dimensions of $U \cap W$.

9. Suppose U and W are 2-dimensional subspaces of \mathbb{R}^2 . Show that $U \cap W \neq \{0\}$. In particular, find the possible dimensions of $U \cap W$. Interpret the results geometrically.

10. If \mathcal{M} and \mathcal{N} are subspaces of \mathbb{R}^n , show that

$$(\mathcal{M} + \mathcal{N})^\perp = \mathcal{M}^\perp \cap \mathcal{N}^\perp$$

and

$$(\mathcal{M} \cap \mathcal{N})^\perp = \mathcal{M}^\perp + \mathcal{N}^\perp,$$