

Homework #1

Due: Wednesday, September 8, 1998

1. Show that, in any vector space, the null-vector is unique. Also, show that to every vector x , there corresponds a unique vector $-x$.

2. Show that the solutions of a homogeneous linear differential equation $y'' + a(x)y' + b(x)y = 0$ form a vector space. What is its dimension?

3. Sometimes a subset of a vector space is itself a vector space (with respect to the linear operations already given). Consider, for example, the vector space \mathbb{R}^3 and the subsets \mathcal{M} of \mathbb{R}^3 consisting of those vectors (ξ_1, ξ_2, ξ_3) for which

(i) $\xi_1 = 0$, (ii) either $\xi_1 = 0$ or $\xi_2 = 0$, (iii) $\xi_1 + \xi_2 = 0$, (iv) $\xi_1 + \xi_2 = 1$.

4. Consider the vector space \mathcal{P} of all polynomials and the subsets \mathcal{M} of \mathcal{P} consisting of those vectors (polynomials) p for which

(i) p has degree 3, (ii) $2p(0) = p(1)$, (iii) $p(t) \geq 0$ for $0 \leq t \leq 1$, (iv) $p(t) = p(1 - t)$ for all t .

Which of these spaces is itself a vector space?

5. (a) Under what conditions on the scalars ξ and η are the vectors $(1, \xi)$ and $(1, \eta)$ in \mathbb{R}^2 linearly dependent?

(b) Under what conditions on the scalars ξ , η , and ζ are the vectors $(1, \xi, \xi^2)$, $(1, \eta, \eta^2)$, and $(1, \zeta, \zeta^2)$ in \mathbb{R}^3 linearly dependent?

(c) Extra Credit: Guess and prove a generalization of (a) and (b) to \mathbb{R}^n .

6. If, in the set \mathbb{C} of all complex numbers, addition is defined as usual and multiplication of a complex number by a real number is defined as usual, then \mathbb{C} becomes a real vector space. What is its dimension?

7. Consider the system of equations:

$$\begin{array}{rccccrcr} w & + & 2x & + & 3y & - & z & = & 1 \\ 2w & + & 4x & + & 7y & - & 2z & = & b_2 \\ -w & - & 2x & - & 4y & + & z & = & b_3 \end{array}$$

What must b_2 and b_3 satisfy for the system to be consistent? What is the solution in that case?