Solutions to Exercises from

Introduction to Perturbation Methods

by

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Chapter 1
Introduction to Asymptotic Approximations

1.3 Order Symbols

1. a) ii) $\alpha \le 2$, v) $\alpha < 1$
2. c) $f = -g = 1/\varepsilon$ and $\varepsilon_0 = 0$

1.4 Asymptotic Approximations

1. d) $\varphi_6 >> \varphi_4 >> \varphi_3 >> \varphi_2 >> \varphi_5 >> \varphi_1$, e) $\varphi_3 >> \varphi_2 >> \varphi_1 >> \varphi_4$, g) not possible
2. b) $2^{3/2}(1 + 3\varepsilon^2/8)$, c) $\sinh(1) + \varepsilon x\cosh(1)/2 - \varepsilon^2 x^2 e^{-1}/8$, d) $e^x(1 - \varepsilon x^2/2)$, g) $1 + \varepsilon n(n + 1)/2$]
3. b) $f = \varepsilon^{-1}$ and $\varphi = -\varepsilon^{-1}$; must have $f - \varphi = o(1)$; Yes, because given any $\delta > 0$ we can get $|f - \varphi| < \delta|\varphi|$. But $\varphi = O(1)$ $|\varphi| \le K$ and so given any $\delta > 0$ we can get $|f - \varphi| < \delta$
5. b) $f = 1 + \varepsilon^2$, $g = -1 + \varepsilon^2$, $\varphi = 1$ and $\phi = -1$ with $\varepsilon_0 = 0$
9. $[[S]] \sim \frac{c_v}{12}\gamma(\gamma^2 - 1)\varepsilon^3 + O(\varepsilon^4)$

1.5 Asymptotic Solution of Algebraic and Transcendental Equations

1. (c) $-2 + \varepsilon$, $1 + \varepsilon^2$, (e) $1/3 + \varepsilon/81$, $\pm(3/\varepsilon)^{1/2} - 1/6$, (f) $\varepsilon + \varepsilon^5$, $\varepsilon^{-1}(\pm 1 - \varepsilon^2/2)$,
   (g) $\pm(e^{1/2} - 1)^{1/2} - \varepsilon(2 \pm (1 - e^{-1/2})^{-1/2})/8$, (h) $1 - 3^{1/2}\varepsilon/2$, $-1 + \varepsilon/2$,
   (i) setting $f = x - \int_0^{\pi} \exp(\varepsilon\sin(x+s))ds$ then $f(0) < 0 < f(2\pi)$ and $f' > 0$ for $0 < \varepsilon < 1/3$ only one solution ; $x \sim \pi - 2\varepsilon$, (m) $x \sim \pm(-\ln\varepsilon)^{1/2}\left(1 + \varepsilon/(2\ln^2\varepsilon)\right)$,
   (n) $x \sim \varepsilon(1 + \varepsilon\alpha + \alpha(\alpha - 1)\varepsilon^2 + \ldots)$,
   (o) $p(r) \sim p_0 + p_1 r + \ldots$ $x \sim p_0^{-1/3}\varepsilon - p_1\varepsilon^2/(5p_0^{5/3})$,
   (p) $x_l \sim \varepsilon(1 + e\varepsilon + \ldots)$ and $x_r \sim \kappa(\zeta - \zeta^{-1} - 1.5\zeta^{-2} + \ldots)$ where $\zeta > 1$ satisfies $\zeta\exp(-\zeta) = \varepsilon$ $\zeta \sim z_0 + \ln(z_0) + \ln(z_0)/z_0$ where $z_0 = \ln(\varepsilon^{-1})$]
2. a) $x \sim k + \varepsilon x_1$ where $x_1 = (-1)^k 210k^{19}/[(k - 1)!(20 - k)!]$
   b) $\varepsilon < 2.2\text{x}10^{-11}$
3. a) $x \sim -\cos(1) + 2\varepsilon\sin(1)\cos(1) + \ldots$

b) $P \sim 2\pi(1 + 100\varepsilon^2)$

6. b) $E \sim M + \varepsilon\sin(M) + \frac{1}{2}\varepsilon^2\sin(2M) + \frac{1}{8}\varepsilon^3(3\sin(3M) - \sin(M))$

7. a) $k(s) \sim \varepsilon/\beta$

8. $x_s \sim \varepsilon\ln(3/y_0) - \varepsilon^2 y_1/y_0$ where $y_0 = y(0) \approx 1.3$ and $y_1 = \frac{1}{2}\ln(y_0/3)\ln\left(\frac{y_0}{2 - y_0}\right)$

9. a) $\lambda \sim \lambda_0 + \varepsilon\lambda_1$ where λ_0 is an eigenvalue for $\mathbf{A}$ and $\lambda_1 = (\mathbf{x}_0^T\mathbf{D}\mathbf{x}_0)/(\mathbf{x}_0\cdot\mathbf{x}_0)$

b) $\lambda \sim 1 + \varepsilon\lambda_1$ where $\mathbf{D}\mathbf{x} = \lambda_1\mathbf{x}$

10. $\mathbf{A}^{-1} - \varepsilon\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}$

11. a) $\mathbf{A}^\dagger + \varepsilon(\mathbf{K}^{-1}\mathbf{B}^T\mathbf{P} \;\; - \mathbf{A}^\dagger\mathbf{B}\mathbf{A}^\dagger)$ where $\mathbf{K} = \mathbf{A}^T\mathbf{A}$ and $\mathbf{P} \;\; = \mathbf{I} - \mathbf{A}\mathbf{A}^\dagger$

1.6 Introduction to the Asymptotic Solution of Differential Equations

2. a) $\tau_h \sim 2 + 4\varepsilon$

3. b) $y \sim y_0 + \varepsilon y_1$ where $y_0 = -\tau/\alpha + (1 + \alpha)(1 - e^{-\alpha\tau})/\alpha^2$ and

$$\alpha y_1 = \int_0^\tau f(s)ds - e^{-\alpha\tau}\int_0^\tau f(r)e^{\alpha r}dr$$

c) Method 1: if $y_0(\tau_0) = 0$ with $\tau_0 > 0$ and $\alpha > 0$ then i) $y_0(2) > 0$, ii) $\frac{d}{d\alpha}\tau_0 < 0$ if $\tau_0 > 1$, and iii) $\alpha = 1$ $\quad 1.5 < \tau_0 < 1.6$. Thus, $1 < \tau_0 < 2$ $\quad \alpha > 0$; Method 2: $\alpha << 1$ $\quad \tau \sim 2 - 2\alpha/3$ $\quad$ decrease

4. $\lambda \sim n\pi(1 - \kappa_n\varepsilon)$ where $\kappa_n = \int_0^1 \mu(x)\sin^2(n\pi x)dx$

5. a) $y \sim A\cos(n\pi x)$ and $\omega \sim n\pi + \varepsilon^2(2 - \alpha^2 - \alpha^4)\frac{n\pi A^2}{16\alpha^2}$ for $n = 1, 2, 3, ...$

8. a) $\lambda \sim a\lambda_0 + a^2\lambda_1$ where $\lambda_0 = K(0,0)$, $\lambda_1 = \frac{1}{2}[K_x(0,0) + K_s(0,0)]$, and

$y \sim y_0(x) + ay_1(x)$ where $y_0(x) = K(x,0)/K(0,0)$, $y_1 = [K_s(x,0) - 2\lambda_1 y_0(x)]/(2\lambda_0)$

c) λ as in (a) and $\varphi \sim 1 + aK_x(0,0)(\xi - 1/2)/K(0,0)$

9. c) $E_1 = \int_{-\infty}^{\infty} V_1\psi_0^2 dx$ and $E_2 = -\int_{-\infty}^{\infty}\psi_0^{-2}\left(\int_{-\infty}^{x}(V_1 - E_1)\psi_0^2 ds\right)^2 dx$

d) $\psi_0 = A_0\exp(-\frac{1}{2}E_0x^2)$, $A_0 = (\lambda/\pi)^{1/4}$

12. $\partial_t c_0 + f(c_0) = \alpha g(t)$ where $c_0(0) = 0$ and $\alpha = ç\,\partial\Omega_1 ç/ç\,\Omega ç$

13. a) ii) $L\partial_{n*} = \partial_z + \varepsilon(f\partial_z^2 - f_x\partial_x - f_y\partial_y) + O(\varepsilon^2)$ evaluated at $z = 0$

14. b) $\rho \sim \varepsilon^{2n}[(\beta - 1)(1 - s^2)/2]^n$ where $r = 1 + \varepsilon s$

Chapter 2
Matched Asymptotic Expansions

2.2 Introductory Example

1. a) $y \sim ax + (1 - a)(1 - e^{-x/\varepsilon})$

2. a) BL at x=0; $y \sim (3 + x)^{-1/2} - e^{-2\bar{x}}/\sqrt{3}$ where $\bar{x} = x/\varepsilon$

   b) BL at x=0; $y \sim g(x) - g(0)e^{-\bar{x}}$ where $g(x) = 1 - \int_x^1 f(s)ds$ and $\bar{x} = x/\varepsilon$

   c) BL at x=0; $y \sim \varepsilon(1 + 2x + 2e^{-\bar{x}})/3$ where $\bar{x} = x/\varepsilon$

   d) BL at x = 0; $y \sim \text{sech}(1)[-\sinh(1) - e^{-x} + (1 + e)e^{-\bar{x}}]$ where $\bar{x} = x/\varepsilon$

   e) $y \sim -3x + 4(5 - 3e^{-4\bar{x}})/(5 + 3e^{-4\bar{x}})$ where $\bar{x} = x/\varepsilon^{1/2}$

   g) $y \sim (1 + 7x^2)^{1/3} - 2^{1/2}3/[\sinh(z) + 2^{1/2}]$ where $z = 3^{1/2}x/\varepsilon + \text{arcsinh}(2^{3/2})$

8. a) $y \sim \beta F(x,1) - \int_x^1 F(x,r)g(r)dr + e^{-p(0)x/\varepsilon}[\alpha - \beta F(0,1) + \int_0^1 F(0,r)g(r)dr\,]$ where $g(r) = f(r)/p(r)$ and $F(x,r) = \exp(\int_x^r q(s)/p(s)ds)$

9. a) $y \sim 1 + ae^{\sqrt{x}} - \kappa^{-1}(1 + a)\int_{\bar{x}}^{\infty} e^{-s^{3/2}}ds$ where $\bar{x} = x/\varepsilon^{2/3}$ $a = 2e^{-1}$, and $\kappa = \frac{2}{3}\Gamma(\frac{2}{3})$

11. a) $y \sim f(x) - f(0)\exp(-\bar{x}^2/2)$ where $\bar{x} = x/\varepsilon^{1/2}$

    b) $y \sim f(x) - f(0)\exp(-q(0)\bar{x}^2/2)$ where $\bar{x} = x/\varepsilon^{1/2}$

12. a) $y \sim \gamma(\gamma - 1)^{-1}[f(t) - f(0)e^{-\bar{t}}] + g(\lambda)e^{-\bar{t}} + \gamma[g_0 - (\gamma - 1)^{-1}f(0)][e^{\gamma^{-1}(1-\gamma)\bar{t}} - e^{-\bar{t}}]$ where $\bar{t} = t/\varepsilon$

13. $\partial_t c_0 + f(c_0) = \alpha g(t)$ where $c_0(0) = |\Omega|^{-1}\int_\Omega h(\mathbf{x})dV$ and $\alpha = |\partial\Omega_1|/|\Omega|$

15. a) $y \sim x + 12z/(1 + z)^2$ where $z = e^{2^{1/2}x/\varepsilon}$

2.3 Examples With Multiple Boundary Layers

1. a) BL at x=0,1; $y \sim 1 - x - \exp(-x/\varepsilon^{1/2}) - \exp((x-1)/\varepsilon^{1/2})$

   b) BL at x=0,1; $u \sim -f(x) + f(0)e^{-\bar{x}/E(0)^{1/2}} + f(1)e^{\tilde{x}/E(1)^{1/2}}$ where $\bar{x} = x/\varepsilon^{1/2}$

   c) BL at x=1; $y \sim (2 - e^{2x})/(2 + e^{2x}) + (1 - A)e^{\bar{x}}$ where $\bar{x} = (x-1)/\varepsilon$ and $A = (2 - e^2)/(2 + e^2)$

   d) BL at x=0,1; $y \sim -e^x f(x) + (f(0) + 1)e^{-\bar{x}} + (ef(1) - 1)e^{\tilde{x}e^{1/2}}$

   e) BL at x=0,1; $y_0 = \sqrt{1 + x^2}$

   g) BL at x=0,1; $y \sim e^x - e^{-\bar{x}} + (4 - e)e^{\tilde{x}}$ where $\bar{x} = x/\varepsilon^{5/4}$

   h) BL at x=1; $y \sim \sqrt{7} - \sqrt{9 - 2x} + Y_0(\bar{x})$ where

$$\int_{-2}^{Y_0} \frac{dr}{A - r^2/2 + r^3/3} = \bar{x} \quad \text{for } A = (-10 + 7\sqrt{7})/3$$

   i) BL at x=1

   j) BL at x=0,1 & 2nd term for matching; $y \sim -x + 2e^{-4\bar{x}} + 4e^{r\tilde{x}}$ where $r = -1 + \sqrt{5}$, $\bar{x} = x/\varepsilon$ and $\tilde{x} = (x-1)/\varepsilon$

   k) BL at x=1; $y \sim Y(\bar{x})$ where $\bar{x} = (x-1)/\varepsilon^{1/2}$ and

$$\int_{Y}^{2} \frac{dr}{\sqrt{2(r - \ln(r) - 1)}} = -\bar{x}$$

l) BL at x=0,1; $y \sim e^x + Y(\bar{x}) - 1 + Z(\tilde{x}) - e^1$ where

$$\int_{2}^{Y} \frac{dr}{r\sqrt{2(r - \ln(r) - 1)}} = -\bar{x}$$

m) BL at x=1; $y \sim y_0(x) + Y_0(\bar{x}) - \alpha$ where $\frac{1}{3}y_0^3 - y_0 = 6 - kx$ for $y_0(0) = 3$, $\alpha = y_0(1)$, $\bar{x} = (x-1)/\varepsilon$, and

$$\int_2^{Y_0} \frac{4dr}{(r^2 - \alpha^2)(r^2 + \alpha^2 + 2)} = \bar{x}$$

2. BL at $x = 0,1$; $y \sim \kappa_0^{-1}\left(1 - e^{-(x+1)/\varepsilon} - e^{(x-1)/\varepsilon}\right)$ and $\kappa \sim \kappa_0 + \ldots$ $\kappa_0 = 2^{1/3}$

3. a) $y \sim H(0)/H(x) + Y_0(\bar{x}) - H(0)/H(1)$ where $\bar{x} = (x - 1)/\varepsilon$ and

$$\bar{x}/H(1) = H(1)(Y_0 - 1) + H(0)\ln\left|\frac{H(0) - H(1)Y_0}{H(0) - H(1)}\right|$$

4. b) $\lambda \sim n^2\pi^2(1 + 4\varepsilon + (12 + n^2\pi^2)\varepsilon^2)$

5. $y \sim -f(x)/q(x) + (\alpha + f_0/q_0)e^{r_0 x/\varepsilon} + (\beta + f_1/q_1)e^{r_1(x-1)/\varepsilon}$ where $f_0 = f(0)$, etc. and $r_0 = -p_0 - \sqrt{p_0^2 + q_0}$ and $r_1 = -p_1 + \sqrt{p_1^2 + q_1}$

6. a) $\gamma = (\beta - \alpha)/2$

   b) $\varphi = \varepsilon\alpha + \varepsilon^{1/2}\gamma\Phi(x,\varepsilon)$

   c) outer: $\varphi \sim \varepsilon\alpha^{1/(2k+1)}$; bdy layer at $x = 0$: $\Phi \sim -\varepsilon^{\kappa}/\left(B + k\bar{x}/(k+1)^{1/2}\right)^{1/k}$ where $\kappa = (2k + 1)/(2k + 2)$, $B = \left(-\gamma(k + 1)^{1/2}\right)^{-k/(k+1)}$ and $\bar{x} = x/\varepsilon^{\kappa}$; solution symmetric about $x = 1/2$

   d) it's necessary to find the second term in the boundary layer to be able to match]

   e) $\gamma = (\beta - \alpha|\Omega|)/|\partial\Omega|$

2.4 Interior Layers

1. a) IL at x=1/2; $y \sim \text{erf}\left((x - 1/2)/\sqrt{2\varepsilon}\right)$

   b) IL at x=0; $y \sim -2/x$ for $x \neq 0$ and $Y \sim 4\varepsilon^{-1/2}\bar{x}M(1,3/2,-\bar{x}^2)$ where $\bar{x}=x/\varepsilon^{1/2}$

   c) for $0 \le x \le \frac{7}{12}$, $y \sim (\frac{5}{3} - x)^{-1} - 2B\cdot De^{B\bar{x}}/(1 + De^{B\bar{x}})$ and for $\frac{7}{12} \le x \le 1$, $y \sim -(\frac{1}{2} + x)^{-1} + 2B/(1 + De^{B\bar{x}})$ where $B = \frac{12}{13}$, $\bar{x} = (x - \frac{7}{12})/\varepsilon$

   d) BL at $x = 0$ and IL at $x = 3/4$; for $0 \le x \le 3/4$, $y \sim Y_0 - 3\exp(-3\bar{x}/16)$ and for $3/4 \le x \le 1$, $y \sim Y_0 + [\,(x - 1/4)^{1/2} - 1/\sqrt{2}\,]/(x - 3/4)^{3/2}$ where
   $Y_0 = 1 + \kappa[\,\Gamma(-1/4)M(3/4, 1/2, -\tilde{x}^2/4) + \Gamma(1/4)M(5/4, 3/2, -\tilde{x}^2/4)\,]$
   Also, $\kappa^{-1} = 32\varepsilon^{3/4}(3\pi)^{1/2}$, $\tilde{x} = (x - 3/4)/\varepsilon^{1/2}$ and $\bar{x} = x/\varepsilon$

e) IL at $x = \frac{1}{2}$; for $1/2 \le x \le 1$, $y \sim y_0(x) + Y_0(\bar{x}) - \alpha$ where $y_0 - \frac{1}{3} y_0{}^3 = x/2 + 1/6$ for $-1 \le y_0 \le -\alpha$, $\alpha = y_0(1/2+)$, $\bar{x} = (x - 1/2)/\varepsilon$, and

$$\int_0^{Y_0} \frac{4dr}{(r^2 - \alpha^2)(r^2 + \alpha^2 + 2)} = \bar{x}$$

f) IL at $x_0 = 1/2$

2. $y \sim 1 + (1 - x^2)^{1/2}$ for $0 \le x < x_s = \sqrt{3}/2$ and $y \sim 0$ for $x_s < x < 1$; near $x = 1$, $y \sim Y_0$ where $(2 - 3/Y_0)\exp(3/(2Y_0)) = \frac{7}{2}\exp(3(\bar{x} - 1)/4)$; for x near x_s , $y \sim Z_0$ where $2/Z_0 + \frac{4}{3}\ln[\,(3/2 - Z_0)/Z_0\,] = \tilde{x} + B$ for $\tilde{x} = (x - x_s)/\varepsilon$

3. b) BL at $x = 0,1$; $y \sim k(2x - 1) + (-3 - k)e^{\tilde{x}} - ke^{-\bar{x}}$ where $\bar{x} = x/\varepsilon$ and $\tilde{x} = (x - 1)/\varepsilon$

4. a) outer: $y \sim \frac{1}{2}\left(k + (k^2 + 4\alpha)^{1/2}\right)$ for $k = 1 - \alpha - x/2$; bdy layer: $\bar{x} = (x - 1)/\varepsilon$, $4a = 1 - 2\alpha + (1 + 12\alpha + 4\alpha^2)^{1/2}$, $4b = -1 + 2\alpha + (1 + 12\alpha + 4\alpha^2)^{1/2}$, and $0 \le Y \le a$ satisfies

$$\left(\frac{a - Y}{a}\right)^a \left(\frac{b + Y}{b}\right)^b = e^{2(a+b)\bar{x}}$$

7. c) $y = -2 + x/4 + \lambda \int_0^x e^{-2rg(r)/\varepsilon} dr$ where $g(r) = (r - a)(r - b) - r^2/3 + ab$ and λ is an ε dependent constant that makes $y(1) = 3$. The desired conclusion comes from the fact that $\lambda \to 0$ if $g(r) < 0$ anywhere in the interval $0 \le r \le 1$. If $\lambda \to 0$ then the layer is at $x = b$.

8. a) $T = T_\infty$

b) $T \sim 1 - \varepsilon \ln(1 - t)$

c) $T \sim T_\infty - \varepsilon T_1$ where $\mu = \varepsilon^{-n}\exp\left(-(T_\infty - 1)/(\varepsilon T_\infty)\right)$ and

$$\tau + \tau_0 = -\int_1^{T_1} r^{-n}\exp(r/T_\infty)dr$$

2.5 Corner Layers

1. a) CL at x=1/2; $y_{outer} \sim \frac{1}{2} + |x - \frac{1}{2}|$ and $y_{inner} \sim \frac{1}{2} + \varepsilon[2\ln(1 + e^{\bar{x}}) - \bar{x}]$ for $\bar{x} = (x - \frac{1}{2})/\varepsilon$

b) CL at x=1/3; $y_{outer} \sim -1 + 3|x - \frac{1}{3}|$

c) CL at x=0; $y_R \sim -xe^{-x}$, $y_L \sim x(2e - e^{-x})$

d) BL at x=0 and CL at x=5/8; $y_R = 0$ for $0 < x \le 5/8$ and $y_L = 1 - \sqrt{9/4 - 2x}$ for $5/8 \le x \le 1$

e) BL at x=0 and CL at x=1/2; $y_R = 1/2$ for $0 < x \le 1/2$ and $y_L = x$ for $1/2 \le x \le 1$; BL at x=0

2. $Y_{CL} \sim 1 + \varepsilon^{\gamma}\alpha_0\big(M(-\gamma, 1/2, -(b - a)\bar{x}^2/2) - \kappa\bar{x}M(1/2 - \gamma, 3/2, -(b - a)\bar{x}^2/2) \big)$ where $\gamma = \frac{a}{2(b-a)}$ and $\kappa = \sqrt{2(b - a)}\ \Gamma(\eta + 1/2)/\Gamma(\eta)$ for $\eta = \frac{b}{2(b-a)}$

3. BL at $x = 0,1$ and a CL at $x = x_s$ where $x_s \sim \varepsilon^{1/2}\ln(3/y_0(0))$; $y \sim y_0(x) + 3e^{\tilde{x}}$ for $x_s < x \le 1$ and $y \sim 3e^{-\bar{x}}$ for $0 \le x < x_s$ where $\tilde{x} = (x - 1)/\varepsilon^{1/2}$, $\bar{x} = x/\varepsilon^{1/2}$ and

$$\int_{y_0}^{2} \frac{dr}{\operatorname{arctanh}(r - 1)} = 1 - x .$$

4. a) boundary layers at $x = 0, 1$ and a corner layer at $x = 1/3$

b) there is an interior layer at $x = \frac{1}{3}\sqrt{2} - 3/8$ and at $x = 11/8 - \frac{1}{3}\sqrt{2}$

c) interior layer at $x = \frac{1}{3}\sqrt{2} - 3/8$, a corner layer at $x = 1/3$, and a boundary layer at $x = 1$

2.6 Partial Differential Equations

4. a) $u \sim h(x + s) + [g(t) - h(s)]\exp[-xv(t)/\varepsilon]$ where $s = \int_0^t v(\tau)d\tau$

b) the 2nd term in the outer expansion is $u_1(x, t) = th''(x + s)$

7. b) $X \sim s(t) + 2\varepsilon\ln(B)/(u_0^{+} - u_0^{-})$

8. a) outer layers ($x \neq 0$): $u \sim \varphi(x - \alpha t)\exp(-\frac{\beta}{2\alpha}(x + \alpha t))$ and shock layer:

$u \sim \frac{1}{2} e^{-\beta t}[\ \varphi(0^+)\mathrm{erfc}(-z) + \varphi(0^-)\mathrm{erfc}(z)\ ]$ where $z = \dfrac{x - \alpha t}{2(\varepsilon t)^{1/2}}$

9. $u_0 = \varphi\big(x - f(u_0)t\big)$ and $\partial_{\bar{x}} U_0 = F(U_0) - s'(t)U_0 + A(t)$ where $F' = f$, $s' = \big(F(u_0^+) - F(u_0^-)\big)/(u_0^+ - u_0^-)$ and $A = \big(u_0^- F(u_0^+) - u_0^+ F(u_0^-)\big)/(u_0^+ - u_0^-)$

12. $c \sim \frac{1}{r+\mu}\,\mathrm{erfc}\Big(\dfrac{x}{2t^{1/2}}\Big) + \Big(1 - \frac{1}{r+\mu}\Big)\mathrm{erfc}\Big(\dfrac{x}{2\varepsilon}\Big(\dfrac{r+\mu}{\mu t}\Big)^{1/2}\Big)$

2.7 Difference Equations

4. a) $y_n \sim A r_+^{\,n} + B r_-^{\,n}$ where $r_\pm = [-\beta \pm (\beta^2 + \alpha^2)^{1/2}]/\alpha$

b) $\alpha = hp_n/2$, $\beta = h^2 q_n/2$; the 2nd derivative doesn't contribute

Chapter 3
Multiple Scales

3.2 Introductory Example

1. a) $y \sim 2\sin(t)/(4 + 3\varepsilon t)^{1/2}$

d) $y \sim \varepsilon^{1/2}\exp(-\alpha t/2)\sin(t/\varepsilon^{1/2})$

3. $\theta \sim \varepsilon\cos\big((1 - \varepsilon^2/16)t\big)$

5. a) $y \sim A(\tau)\cos(t) + B(\tau)\sin(t)$ where $A(0) = 0$, $B(0) = 1$,

$$A' = -\frac{1}{2\pi}\int_0^{2\pi} F(-A\sin t + B\cos t)\sin t\,dt \quad \text{and} \quad B' = -\frac{1}{2\pi}\int_0^{2\pi} F(-A\sin t + B\cos t)\cos t\,dt$$

b) $A = 0$ and $B = 3\pi/(3\pi + 4\tau)$

7. $\varphi \sim \alpha\big[1 - \sqrt{2}\exp\big(-(1 + \gamma)\varepsilon t/2\big)\sin(t + \pi/4)\big]$

9. a) $\varphi \sim A(x,t)\exp[-iv(x)\sin(t/\varepsilon)]$ where $2iA_t + 2A_{xx} = v_x{}^2 A$

10. b) $\mathbf{q} = \mathbf{q}_0\exp(i\omega t_1)$ where $\omega = (\beta\gamma - \alpha^2)^{1/2}$ and $\mathbf{q}_0 = (\beta, i\omega - \alpha)^T$

c) $\mathbf{p} = \alpha_0\mathbf{p}_0\exp(i\omega t_1) + cc$ where $\mathbf{p}_0 = (\gamma, i\omega + \alpha)^T$

3.3 Slowly Varying Coefficients

2. $y \sim D(\varepsilon t)^{-1/4}\big(\beta D(0)^{3/4}\sin\tau + \alpha D(0)^{1/4}\cos\tau\big)$ where $\tau = \int_0^t D(\varepsilon s)^{-1/2}ds$

4. The higher order terms for the stability boundaries can be found in Abramowitz and Stegun (1972), pg 724. Note, using their notation, $a = 4\lambda$ and $q = \varepsilon/4$.

7. b) $s = g(x)/g(1)$ where $g(x) = \int_0^x [1 + \varepsilon\mu(r)]^{1/2}dr$ and $\nu = \lambda g(1)$

c) $h \sim \mu'(s)/2$ $\quad \nu \sim n\pi - \varepsilon v_1$ where $4v_1 = \int_0^1 \mu'(s)\sin(2n\pi s)ds$

d) $\nu_1 = 0$ $\quad \lambda \sim \lambda_0 + \varepsilon\lambda_1$ where $\lambda_0 = n\pi$ and $\lambda_1 = v_1 - v_0/2 = -n\pi/2$. Also, $Y \sim A(1 - \varepsilon s/4)\sin(n\pi s) + B\varepsilon\sin(n\pi s)$ and $s \sim x[1 + \varepsilon(x - 1)/4]$

3.4 Forced Motion Near Resonance

1. a) $u \sim \alpha + (\alpha_0 - \alpha)\sin(\theta - \varepsilon\alpha\theta + \pi/2)$

b) $2\pi(1 + \alpha^2\varepsilon)$

c) $\Delta\varphi \approx 42.98''$

2. b) $A_\infty^2[9\lambda^2 + (6\omega - \frac{3\kappa}{4}A_\infty^2)^2] = 1 \qquad A_M = (3\lambda)^{-1}$

3. a) $y \sim A(\varepsilon t)\cos(t + \theta_0)$ where $A = 2$ or $A = 2[c/(c + 4\exp(-\varepsilon t))]^{1/2}$

4. a) $\theta \sim \varepsilon^2(\sin\omega t - \omega\sin t)/(1 - \omega^2) + O(\varepsilon^8)$

b) $2A' = -\cos(\theta - \omega\varepsilon t)$ and $A_\infty^3 + 16\omega A_\infty \pm 8 = 0$

5. $y \sim \varepsilon^{1/3}A(\varepsilon t)\cos[\kappa t + \theta(\varepsilon t)]$ where $A_\infty(\frac{3\lambda}{8}A_\infty^2 - 2\omega\kappa) = 1$

6. a) $y \sim \varepsilon^{1/3}A(\varepsilon^{2/3}t)\cos[t + \theta(\varepsilon^{2/3}t)]$ where $A_\infty^2[\alpha^2 + (2\omega + A_\infty^2)^2] = 1$

b) yes

7. a) $y \sim f(v)(1 - e^{-\gamma\varepsilon t}\cos(t))$ where $\gamma = \beta + a(v^2 - 1)$

b) $y \sim \varepsilon^{-1}A(\varepsilon t)\cos(t + \theta_0)$ where $2A' + [\beta + a(v^2 - 1 + \frac{1}{4}A^2)]A = 0$

Other references: Derjaguin, et al. (1957) and Vatta (1979)

10. a) $\theta \sim \varepsilon^{1/2}A(\varepsilon t)\cos[t + \varphi(\varepsilon t)]$ where $8A' = [\alpha\sin(2\varphi) - 4\mu]A$ and if $A \neq 0$ then $16\varphi' = 2\alpha\cos(2\varphi) - A^2 - 4\alpha$. Note, $\alpha < 4\mu \quad A \to 0$

3.6 Introduction to Partial Differential Equations

5. a) $u(x,t) \sim \sum_{n=1}^{\infty} \beta_n e^{-\varepsilon t/2}\cos((\lambda_n^2 - \varepsilon/2)t)\sin\lambda_n x$

8. b) $u \sim \varepsilon(1 - x)U(\tau)$ and $\rho \sim \left(1 - \int_0^\tau U(\sigma)d\sigma\right)^{-1}$

3.7 Linear Wave Propagation

1. $u \sim [c(0)/c(\varepsilon t)]^{1/2}[\, f(x - \int_0^t c(\varepsilon\tau)d\tau) + f(x + \int_0^t c(\varepsilon\tau)d\tau)\,]/2$

2. $u \sim \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} f(x - t + 2r(\varepsilon t)^{1/2})e^{-r^2}dr$

3. a) $p \sim [F(x - t) + G(x + t)][A(0)/A(\varepsilon x)]^{1/2}$
4. characteristics satisfy $x_t = y$, $y_t = -f(t)x$ $\quad x'' + (a - \varepsilon\cos(t))x = 0$ which is Mathieus' eq

3.8 Nonlinear Waves

2. b) $u \sim \varepsilon\{B_0 + A_0\cos[\theta + \varphi(\tau)]\}$ where $\theta = kx - \omega t$ and $\tau = \varepsilon^\gamma t$; (i) $B_0 \neq 0$ $\quad \gamma = 1$, $\varphi = \varphi_0 - \alpha\kappa\tau B_0$, and (ii) $B_0 = 0$ $\quad \gamma = 2$, $\varphi = \varphi_0 + C\tau$ for $C = \alpha kc_0 - \alpha^2 A_0{}^2/(24\beta k)$ where c_0 is an arbitrary constant
c) $u \sim \varepsilon[\mu + A(r)\cos[\theta + \alpha\varepsilon\mu(x - t)/(3\beta k) + \varphi_0(r)]]$ where $r = \varepsilon x - (1 - 3\beta k^2)\varepsilon t$; the case of when $\mu = 0$ is considered in Schoombie (1992)
4. b) $u \sim [1 + \exp(\lambda(x + t) - \varepsilon t)]^{-1}$
6. a) $u_0 = U_0(t_1 + \theta(x, t_2))$ where

$$U_0(s) = \frac{1}{1 + e^{-s}}$$

and $\theta = \frac{1}{2}\ln(1 + 4\varepsilon\lambda t) - \frac{1}{1 + 4\varepsilon\lambda t}\left[\varepsilon\lambda^2 t(x_1 - x_0)^2 - \lambda(x - x_0)(x - x_1)\right]$
8. b) $u \sim u_0(\theta, x_2)$, for $\theta = t - x$ and $x_2 = \varepsilon x$, where $u_0 = 0$ if $\theta < 0$ otherwise $\partial_\theta u_0 = g'(s)$ where $s = s(\theta, x_2)$ is the non-negative solution of

$$\theta = s - \tfrac{1}{2}x_2 f(-g'(s)) .$$

Thus, for $\theta \geq 0$, $u_0 = g(s) + \frac{1}{2}x_2[g'(s)f(-g'(s)) + F(-g'(s))]$ where $F'(\sigma) = f(\sigma)$ with $F(0) = 0$.
9. a) $s \sim \frac{1}{2}[F(\varphi_1) + F(\varphi_2)]$ where $\varphi_1 = x - t - \varepsilon tF(\varphi_1)/2$ and $\varphi_2 = x + t + \varepsilon tF(\varphi_2)/2$
10. a) $\rho \sim 1 + \varepsilon f(x - \lambda t) + 2\varepsilon^2\lambda tf(x - \lambda t)f'(x - \lambda t)$ where $\lambda = \alpha + \beta$

3.9 Difference Equations

1. a) $y_n \sim Ae^{2\varepsilon n}$

3. a) $y_n \sim \alpha^n \bar{y}_0(s)$ where $\bar{y}_0{}' = g(\bar{y}_0)$

4. from (a), $\theta' = \frac{3}{8}A^2$; from (i), $\theta' \sim \frac{3}{8}A^2(1 + \frac{1}{8}h^2)$; from (ii), $\theta' \sim \frac{3}{8}A^2(1 - \frac{5}{8}h^2)$;

   from (iii), $\theta' \sim \frac{3}{8}A^2(1 - \frac{1}{8}h^2)$

5. a) $\omega^2 = \omega_d{}^2 + 4\sin^2(k/2)$

b) $u_n(t) \sim \varepsilon A(\bar{t})\cos(\, kn - \omega t + \varphi(\bar{t})\,)$

6. a) $g_n \sim G_0(\varepsilon n)$ where $G_0{}'(s) = \gamma G_0(1 - G_0)(\alpha + \beta G_0)$ where $\gamma = N^{-1}\sum_{j=1}^{N} f(j)$

b) $g_n \sim 0, 1, -\alpha/\beta$; $0 < g_\infty < 1$ if $\alpha > 0$ and $\beta < 0$ with $0 < g_1 < 1$

Chapter 4
The WKB and Related Methods

4.2 Introductory Example

1. c) $y_{wkb} \sim A[\exp(-e^x) - \exp(e^x - 2 - x/\varepsilon)]$ where
   $A = (\exp(-e) - \exp(e - 2 - 1/\varepsilon))^{-1}$ and $y_{com} \sim \exp(e - e^x) - \exp(-1 + e - x/\varepsilon)$

3. d) $y \sim q^{-1/4}(Ae^{\eta/\varepsilon} + Be^{-\eta/\varepsilon})\exp(-\frac{1}{2}\int^x f(r)dr)$ where $\eta = \int^x \sqrt{q(r)}dr$

4. c) $y \sim h^{-1/2}[\, a_0\exp(\frac{1}{2\varepsilon}\int^x (-p + h + \frac{\varepsilon p'}{h})ds + b_0\exp(-\frac{1}{2\varepsilon}\int^x (p + h + \frac{\varepsilon p'}{h})ds\,]$

   where $h = (p^2 - 4q)^{1/2}$

7. c) $w \sim g(x)[\,(a + b/n)\sin(\lambda_n\theta) - \frac{a\cos(\lambda_n\theta)}{2\lambda_n}\int_0^x (pq)^{-1/2}(f - 2q\lambda_0\lambda_2 - pg''/g)ds\,]$ where

   a,b are arbitrary constants, $\theta = \int_0^x (q/p)^{1/2}ds$ and $\lambda_0 = \pi/\kappa$

e) $\lambda \sim n\left(1 - \frac{49}{72(n\pi)^2}\right)$

8. $y_\pm \sim q^{-1/4}\exp[\,\frac{1}{2\varepsilon}\int^x (\pm(-4q + \varepsilon p^2)^{1/2} - \varepsilon^{1/2}p)dx\,]$

10. a) $y_\pm \sim \kappa_\pm x^{\pm\nu}[1 \mp \frac{1}{4\nu}(\alpha_\pm + x^2) + \ldots$

b) $J_\nu(x) \sim (2\pi\nu)^{-1/2}\left(\frac{ex}{2\nu}\right)^\nu [1 - \frac{1}{12\nu}(1 + 3x^2)]$

11. a) $y \sim \varepsilon e^{t/2}\{\sin[\alpha(1 - e^{-t})] + \frac{1}{8}\varepsilon(e^t - 1)\cos[\alpha(1 - e^{-t})]\}$ where $\alpha = 1/\varepsilon$

b) two principal shortcomings are: (1) differences in zeros --- this is a relatively minor problem as discussed in Section 1.4, and (2) unboundedness of second term for large t --- in particular, we need $\varepsilon e^t << 1$. The latter problem is due to a turning point at $t = \infty$.

c) $y = \frac{\pi}{2}[Y_0(\alpha)J_0(\beta) - J_0(\alpha)Y_0(\beta)]$ where $\alpha = 1/\varepsilon$ and $\beta = \alpha e^{-t}$

12. a) $\varepsilon\beta R'' + (-1 + \lambda\beta)R' = 0$ and $\varepsilon\beta R'(0) - R(0) = 0$ $\qquad R = 1 - re^{-kx/\varepsilon}$ where $r = 1/(\beta\lambda)$ and $k = (-1 + \lambda\beta)/\beta$

b) $R \sim 1 + Ae^{-\theta(x)/\varepsilon}(1 - \lambda\beta)^{-1}$ where $\theta = \lambda x - \int_0^x \beta^{-1}(s)ds$ and

$A = 1 - (\lambda\beta_0)^{-1}$; the conditions are that $\theta \to \infty$ as $x \to \infty$ and $\lambda\beta > 0$ for $0 < x < \infty$

13. $P \sim A(x,\varepsilon)\exp\left(\theta(x)/\varepsilon + \eta\mu_0 x(e^{\theta_x} - 1) - \int_0^\eta \lambda(s)ds \right)$ where $\theta_x = \ln(\xi)$ and $\xi(x)$

satisfies $1 = \mu_0 x \int_0^\infty \exp\left(\mu_0 x(\xi - 1)\eta - \int_0^h \lambda(s)ds \right)d\eta$. Also, A is determined from the $O(\varepsilon)$ term from the $\eta = 0$ condition

4.3 Turning Points

3. b) $E = \varepsilon(2n + 1)\mu^{1/2}$

d) $E \sim (\alpha\mu\varepsilon^m N^2)^{1/\gamma}(1 + \beta/N^2)$ where $4\alpha = \pi\left(\Gamma(\frac{3m+2}{2m})/\Gamma(\frac{m+1}{m})\right)^2$ and $3\pi(m + 2)^2\beta = 4m(m - 1)\cot(\pi/m)$

4. b) the approximation for $\psi(x)$ is given by (4.44) for $x < a$, and for $x > b$ one finds

$\Psi \sim 2a_R |q(x)|^{-1/4} e^{-\varphi/\varepsilon}\exp\left(i(-Et - \bar{\theta}/\varepsilon - \pi/4)\right)$ where $q = V(x) - E$,

$$\varphi = \int_a^b \sqrt{q(s)}\, ds \quad \text{and} \quad \bar{\theta} = \int_x^b \sqrt{q(s)}\, ds$$

8. b) balancing $\alpha = 1/2$ and $\beta = 1/4$

4.4 Wave Propagation and Energy Methods

4. b) $E = \frac{1}{2} Du_{xx}^2 + \frac{1}{2}\mu u_t^2$, $S = -u_{xt}Du_{xx} + u_t\partial_x(Du_x)$, $\Phi = 0$

4.5 Wave Propagation and Slender Body Approximations

1. a) for the n*th* mode the general solution has the same form as in (4.84) but

$$u_L = \frac{\sin(\lambda_n y)}{\sqrt{\theta_x}} \left\{ a_L e^{i[\omega t - \theta(x)/\varepsilon] + \zeta(x)} + b_L e^{i[\omega t + \theta(x)/\varepsilon] - \zeta(x)} \right\}$$

where $\theta(x)$ is given in (4.87) and $\zeta(x) = \frac{\omega}{2}\int_0^x \alpha(x)(\omega^2\mu^2 - \lambda_n)^{-1/2}\,dx$

4.6 Ray Methods

1. a) It's not hard to show that $a_i\mathbf{d}_i + a_r\mathbf{d}_r + a_t\mathbf{d}_t = \mathbf{0}$ where a_i, a_r, a_t are not all zero.

b) Using the $\mathbf{t}$, $\mathbf{n}$, $\mathbf{t}\times\mathbf{n}$ coordinate system, where $\mathbf{t}$ is the unit tangent to S in the plane of incidence which satisfies $\mathbf{t}\cdot\mathbf{d}_i > 0$, then $\theta_I = \mu_+\mathbf{d}_i\cdot\mathbf{x}$, $\theta_R = \mu_+\mathbf{d}_r\cdot\mathbf{x}$, and $\theta_T = \mu_-\mathbf{d}_t\cdot\mathbf{x}$

$\mathbf{u}_I = (\alpha, -\beta, 0)$, $\mathbf{u}_R = (\alpha, \beta, 0)$ and $\mathbf{u}_T = (\kappa\alpha, -(1-\kappa^2\alpha^2)^{1/2}, 0)$ where $\kappa = \mu_+/\mu_-$ and $\beta = (1-\alpha^2)^{1/2}$

3. a) $D(\ \theta\cdot\ \theta)^2 = \mu$

4. b) SIAM Appl Math, v16, 1968, 783-807

5. a) $\alpha = \mu(-1/2)\sin(\phi)$ where $z_R = \min\{0$, roots of $\mu(z) = \alpha$ for $z > -1/2\ \}$

c) $v_0 = [\tan\phi/(\mu J)]^{1/2}/(4\pi)$

7. a) μ = constant   rays are straight lines   wave fronts parallel $R_i = \rho_i + s$;
in R^2 , $\rho_2 = \infty$

8. a) (4.103) $\bar{\mathbf{x}}_{ss} = a\bar{\mathbf{x}} - b\bar{\mathbf{x}}_s$ $\mathbf{p}_s = \mathbf{0}$

b) $\kappa = \pm r_0\mu(r_0)\sin(\varphi_0)$ where φ_0 is the initial angle the ray makes with the radius vector

10. c) note $\kappa \approx 0.065r_c$ where r_c is the scale factor (in km) used to nondimensionalize $r^* = r_c r$

4.8 Discrete WKB Method

6. a) $y_n \sim (q_n^2 - 4)^{-1/4}[a_0\exp(\theta_+/\varepsilon) + b_0\exp(\theta_-/\varepsilon)]$ where

$$\theta_\pm = \int^{\varepsilon n} \ln[\tfrac{1}{2}\left(q(\nu) \pm \sqrt{q(\nu)^2 - 4}\right)]d\nu\ .$$

b) $\alpha_{n+1} = (c_n/a_n)\alpha_{n-1}$

7. a) $y_n \sim (1-\nu^2)^{-1/4}\left(a_0\exp(i\theta/\varepsilon) + b_0\exp(-i\theta/\varepsilon)\right)$ where $\nu = \varepsilon n$ and

$\theta = \nu\cos^{-1}(\nu) - (1 - \nu^2)^{1/2}$

b) $a_0 = (\varepsilon/2\pi)^{1/2}\exp(i\pi/4)$ and $b_0 = (\varepsilon/2\pi)^{1/2}\exp(-i\pi/4)$

8. a) $z_n \sim \nu^{1-\alpha/2}(1 + 4\nu)^{-1/2}(ar_n + b/r_n)$ where $\nu = \varepsilon n$ and

$$r_n = \left(\frac{1 + 2\nu + (1 + 4\nu)^{1/2}}{\nu}\right)^{n-\alpha/2} \exp\left(\frac{(1 + 4\nu)^{1/2}}{2\varepsilon}\right)$$

9. b) $\gamma = k - n$ and $R \sim e^{\theta(\kappa)/\varepsilon}[\, R_0(n,\kappa) + \dots \,]$, where $\theta = \kappa + (1 - \kappa)\ln(1 - \kappa)$ and $R_0 = A[(1 - \kappa)e^n]e^{n/2}$.

Chapter 5
The Method of Homogenization

5.2 Introductory Example

2. a) $\partial_x(\bar{D}\partial_x u_0) + g(u_0) = \partial_t u_0 + f_\infty$

8. e) $D^{-1}{}_\infty{}^{-1} = 1/(\varphi_\alpha/D_\alpha + \varphi_\beta/D_\beta)$ which is a volume fraction weighted harmonic mean ; the harmonic mean is obtained only when $\varphi_\alpha = \varphi_\beta = 1/2$

f) the greatest relative error occurs when $\varphi_\alpha = 1/2$ with a relative absolute error of $(D_\alpha - D_\beta)^2/(4D_\alpha D_\beta)$; the error is zero if $\varphi_\alpha = 0$ or if $\varphi_\alpha = 1$

5.4 Porous Flow

3. b) setting $\mathbf{w}_q = (u_q, v_q, w_q)$ for $q = s,f$ $\nabla^2\mathbf{w}_q = \mathbf{0}$ where $\mathbf{w}_q$ is periodic and on the interface $D_s\mathbf{n}\cdot(\mathbf{e}_1 - \nabla_y u_s) = \alpha D_f \mathbf{n}\cdot(\mathbf{e}_1 - \nabla_y u_f)$ etc

4. b) $(\nabla_y^2 - z^2\alpha e^{z\varphi_0})f = z\alpha e^{z\varphi_0}$ where $\mathbf{n}\cdot\nabla_y f = 0$ on $\partial\Omega_f$

e) φ satisfies the Poisson-Boltzmann equation $\nabla_y^2\varphi = z\alpha e^{z\varphi}$ in Ω_f where φ is periodic and $\mathbf{n}\cdot\nabla_y\varphi = \sigma$ on $\partial\Omega_f$.

Chapter 6
Introduction to Bifurcation and Stability

6.2 Introductory Example

1. b) $y_{\pm} = \pm(1 - 4\lambda^2)^{1/2}/3$ for $-1 \le 2\lambda \le 1$ where y_+ stable and y_- unstable

3. a) supercritical pitchfork when $\lambda_n = (n\pi)^2$ and $\theta_n \sim 2(2\varepsilon)^{1/2}\cos(n\pi x)/(n\pi)$

b) $V_1 \sim -2\varepsilon^2/\pi^2$ $\theta_s = 0$ unstable for $\lambda > \pi^2$

c) for θ_n one finds $V_n \sim -2\varepsilon^2/(n\pi)^2$ $V_1 < V_2 < ... < V_n < 0$ and this suggests V_1 is the preferred configuration

4. a) $y = A\sin(n\pi x)$ where $A = 0$ or $A^2 = 2(\lambda - (n\pi)^2)$; pitchfork bifurcation at $(\lambda_p, A_p) = ((n\pi)^2, 0)$ for $n = 1, 2, 3, ...$

b) $V_n = -A_n{}^4/2$

5. a) $\theta = 0$ stable for $0 < \omega < 1$ and $\theta = \pm\arccos(\omega^{-2})$ stable for $1 < \omega < \infty$; $\theta = \pm\pi$ unstable

6. c) $\kappa_0 = 0$ and $0 < \kappa_1 < 1/2$

d) slope for $\Omega = 40$ is 24.34 and slope for $\Omega = 30$ is 20.36 ; $\omega = \varepsilon(1/2 - \kappa_2)^{1/2}\Omega$

7. c) $\omega = 2$ and $v \sim A(\tau)\cos(t + \theta(\tau))$ where $A = A_0(\alpha_0{}^2 + e^{-3\tau})^{1/2}e^{3\tau/4}$ and $\theta = -\pi/4 + \arctan(\alpha_0 e^{3\tau/2})$

6.5 Relaxation Dynamics

3. b) $\ln(y_0) - \frac{1}{2}y_0{}^2 = t + \frac{1}{2}(\ln 3 - 3)$ and the corner is at $t = 1 - \frac{1}{2}\ln 3$, $y = 1$ and $v = -2/3$

d) $T \sim 3 - 2\ln 2 \approx 1.614$

5. a) subcritical saddle-node at $(\lambda_b, g_b) \approx (0.1292, 0.0635)$

6. a) supercritical saddle-node at $(\lambda_0 , y_0) \approx (280.76, 2.360)$ and subcritical saddle-node at $(\lambda_1 , y_1) \approx (336.6, 6.6355)$

c) $\tilde{Y} \sim y_1 - \sigma\varepsilon^{1/3}\big(a_0\mathrm{Ai}'(-\xi\tilde{\tau}) + a_1\mathrm{Bi}'(-\xi\tilde{\tau})\big)/\big(a_0\mathrm{Ai}(-\xi\tilde{\tau}) + a_1\mathrm{Bi}(-\xi\tilde{\tau})\big)$ where $\sigma = \xi/(15y_1 - 74)$ $\xi = [(15y_1 - 74)(y_1 - 1)]^{1/3}$ and $\tilde{\tau} = (\tau - \lambda_1)/\varepsilon^{2/3}$ $\tau_0 = \lambda_1 + (2.3381...)\varepsilon^{2/3}/\xi$

7. a) transcritical bifurcation at (0,0) and a subcritical saddle-node at (9/8, 3/4)

c) $y \sim y_s + \varepsilon^{1/3} A\cos(\omega t + \theta)$ where $A_\infty{}^2\{ \kappa^2\omega^2 + \frac{9}{4} A_\infty{}^4[-1 + 7(1 - 2y_s)^2/\omega^2]^2\} = 1$

6.6 An Example Involving A Nonlinear Partial Differential Equation

1. a) $u_s = \pm\varepsilon^{1/2} A\sin(nx)$ and $\lambda_b = n^2$ where $A = 2/\sqrt{3}$

b) $u_s = 0$ stable for $\lambda < 1$ and $u_s = \pm\varepsilon^{1/2} A\sin(x)$ stable for $0 \le \lambda - 1 << 1$

2. a) $u_s = \varepsilon^\alpha A\sin(n\pi x)$ and $\lambda_b = (n\pi)^2$, where (i) n odd $A = 3/(8n\pi)$ and $\alpha = 1$, and (ii) n even $A^2 = 6/(5\lambda_b)$ and $\alpha = 1/2$

b) stable for $\lambda < \pi^2$

Additional References: Bramson (1983) and Fisher (1937)

3. a) (i) $u_s = 0$ λ , and (ii) $u_s = A_n \sin(n\pi x)$ with $\lambda = \lambda_n(1 + A_n{}^2/8)$ where $\lambda_n = (n\pi)^2$

b) unbuckled state is stable if $\lambda < \pi^2$ and unstable if $\lambda > \pi^2$

6. b) $\kappa = 0$ and stable for $\lambda < 1$

c) $v \sim \varepsilon[B_0 + A_0\cos((1 + \frac{1}{2}\varepsilon)x + \theta_0)]$ and $\kappa \sim \frac{1}{4} A_0{}^2\varepsilon^2$ where $\varepsilon = \lambda - 1$

8. c) $u_0 = (u_i + u_j)/2$, $\alpha_{ij} = -(u_i - u_j)/2$, $\beta_{ij} = (1/8)^{1/2}(u_j - u_i)$, and $\chi_{ij} = 2^{1/2}\mathrm{sgn}(u_j - u_i)(u_1 + u_2 + u_3 - 3u_0)$; see Albano, et al. (1984)

9. b) for $0 \le \lambda < 7$ the steady states are $u_i = \alpha_i x$, for $\alpha_1 < \alpha_2 < \alpha_3$, where u_1, u_3 are stable and u_2 is unstable; for $7 < \lambda$ the steady state u_3 is unique and stable

10. a) $u \sim u_s + \sum v_n e^{int}$ $\lambda < 1$

6.7 Bifurcation of Periodic Solutions

1. a) saddle-node at (−1/4,−1/2) , a Hopf at (0,0) , and a transcritical at (0,−1)

b) $y \sim \varepsilon^{1/2} A\cos(t_1 + \theta)$ where $8A' = A(4 - A^2)$ and $24\theta' = 36 - 31A^2$]

2. a) $y_s = 0$ asy. stable only when $\lambda < 0$

b) $y \sim A(\varepsilon t)\sin(t + \theta_0)$ where $A(\tau) = 1/(\alpha_0\alpha + c\exp(-n\tau))^{1/2n}$

3. b) $y = 0$ unstable; $y = y_0$ stable for $0 < rT < \pi/2$ and for $rT > \pi/2$ there are stable limit cycles

4. b) $y = y_0$ is stable for $\lambda y_0 < 2$ and there's a Hopf bifurcation at $\lambda y_0 = 2$

6.8 Systems of Ordinary Differential Equations

3. a) $r' = -r(r^2 - \lambda(1-\lambda))$ and $\theta' = -1$
4. a) $y = v = (1 - \alpha - \beta + \mu)/2$
5. a) $y = v = 1$ is stable for $\lambda < (\gamma - 1)^{-1}$ and unstable for $\lambda > (\gamma - 1)^{-1}$

b) a Hopf bifurcation takes place

6. a) $x = \gamma/(1-\gamma)$, $n^2 = (1-\gamma)/\alpha$ is stable for $(1-\gamma)^3 < 4\alpha$
7. a) $c_s = T_s\exp(-T_s)$ and $T_s = \mu/\kappa$; steady state is asymptotically stable if $\kappa > (T_s - 1)\exp(-T_s)$

b) supercritical Hopf bifurcation at μ_l and subcritical Hopf bifurcation at μ_r

d) $\mu_l \sim \kappa(1 + e\kappa + e^2\kappa^2 + ...)$ and $\mu_r \sim \kappa(\zeta - \zeta^{-1} - 1.5\zeta^{-2} + ...)$ where $\zeta > 1$ satisfies $\zeta\exp(-\zeta) = \kappa$ $\quad \zeta \sim z_0 + \ln(z_0) + \ln(z_0)/z_0$ where $z_0 = \ln(\kappa^{-1})$

8. a) $x_s = \mu/\alpha$, $y_s = \alpha/\mu$ is stable

b) $x_s = \mu/\alpha$, $y_s = \alpha/\mu$ is stable if $\mu > \mu_c$ and it's unstable if $0 < \mu < \mu_c$,
where $\mu_c = \alpha(1-\alpha)^{1/2}$

9. a) $\theta = z = 0$

b) $\theta = 0$ and $z = \varepsilon\alpha_0\cos(\kappa t)$; the motion is straight up and down; see van der Burgh (1968)

10. a) $\lambda < 0$

b) $k^2 = \lambda - r_0^2$ and $0 < r_0 < \lambda^{1/2}$

11. a) Hopf bifurcation along line $\lambda = -\mu$ for $-1 < \mu < 1$

b) $y \sim \varepsilon^{1/2}y_0(t,\tau)$ and $v \sim \varepsilon^{1/2}v_0(t,\tau)$ where $\tau = \varepsilon t$. As $t \to \infty$, $y_0 \to \sin(\Theta)$ and $v_0 \to (1-\mu^2)^{1/2}\cos(\Theta) + \mu\sin(\Theta)$ where $\Theta = (1 - \varepsilon\mu)(1-\mu^2)^{1/2}\, t + \theta_0$