Owari II. Marching Groups and Bulgarian Solitaire

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Abstract

This note is a continuation of [1]. We establish a duality relation between Bulgarian solitaire, a patience introduced by Gardner, and open owari. These two games give raise to similar periodical structures involving augmented marching groups. We also give a simpler proof of the main theorem of [1] and we consider a generalization of marching groups to closed owaris introduced by Bruhn [3].

1 Introduction

An *open owari* is made of finitely many pebbles distributed into holes placed along an open line oriented from left to right. Every hole is followed by a hole on its right and preceded by a hole on its left. Accordingly there are infinitely many holes. We assume the number of pebbles to be nonnull. Hence there is a leftmost nonempty hole called the *tail* and a rightmost nonempty hole called the *head*. A *derivation* of an open owari consists in scooping the pebbles in the tail and to sow them, one by one, into the subsequent holes. The number of pebbles in a hole is the *weight* of that hole. The *weight sequence* $[w_0, w_1, \dots, w_{l-1}]$ records the weights of the successive holes, starting with the weight w_0 of the tail and finishing with the weight w_{l-1} of the head. An owari with no null entry in its weight sequence is a *queue*. A derivation of a queue gives another queue.

A queue is *periodical* if there is a positive integer p such that we obtain a queue with the same weight sequence after p derivations. The smallest value of p is the *period*. It is easy to verify that a queue has period 1 if and only if its weight sequence is of the form $[n, n-1, \dots, 2, 1]$, for some positive integer n. Such a queue is called a *marching group* after [5] (see also [4]). A sequence of integers satisfies Property **AMG** if it is of the form

$$[n + a_0, n - 1 + a_1, n - 2 + a_2, \dots, 1 + a_{n-1}, \widehat{a_n}],$$

where n is a positive integer, $[a_0, a_1, a_2, \dots, a_{n-1}, a_n]$ is a sequence of integers equal to 0 or 1 and the hat above a_n means that the term is present if $a_n = 1$ and missing if $a_n = 0$. A queue is an *augmented marching group* if its weight sequence satisfies Property **AMG** . The main result of [1] is the following one.

Theorem 1 *A queue is periodical if and only if its weight sequence satisfies Property* **AMG** *.*

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A *Bulgarian solitaire* is made from a deck of cards divided into piles. A *derivation* consists in removing one card in each pile to form a new pile. The *weight sequence* of the Bulgarian solitaire is the sequence of the numbers of cards of its piles arranged in a monotone nonincreasing order. The Bulgarian solitaire is *periodic* if there is a positive integer p such that one retrieves a Bulgarian solitaire with the same weight sequence after performing p derivations. The smallest value of p is the *period*.

Theorem 2 (Brandt[2]) *A Bulgarian solitaire is periodic if and only if its weight sequence satisfies Property* **AMG** *.*

In Section 2 we establish a duality relation between Bulgarian solitaires and a subclass of open owaris that explains why periodic queues and periodic Bulgarian solitaires have the same structure. In Section 3 we simplify the proof given in [1] of Theorem 1. In Section 4 we state a result of Bruhn [3] that generalizes Theorem 1 to closed owaris, where the holes are placed along a closed curve.

2 Bulgarian solitaires, monotone owaris and Ferrer matrices

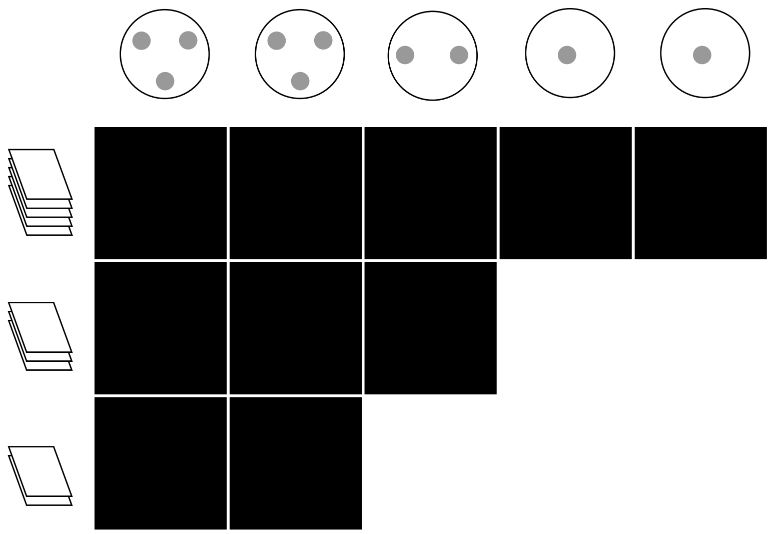

Figure 1: Duality. The Bulgarian solitaire on the left is arranged according to nonincreasing weights, from top to left. By Replacing each pile of cards by the same number of black squares arranged into a row we obtain a Ferrer diagram (which looks like upside down irregular stairs). The nonempty holes of the open owari at the top correspond to the columns of the Ferrer diagram. The weight of a hole is equal to the number of black squares in the corresponding column. This open owari, which has a monotone nonincreasing weight sequence, is the dual of the Bulgarian solitaire.

Let $w = [w_0, w_1, \ldots, w_{l-1}]$ be a sequence of integers. For an index i, such that $0 \le i < l$, the *i-suffix* of w is the subsequence $[w_i, w_{i+1}, \ldots, w_{l-1}]$. The *last peak* of w is the minimal value of i such that the i-suffix is monotone nonincreasing. We note that w is monotone nonincreasing if its last peak is equal to 0. The last peak of an open owari is the last peak of its weight sequence. An open owari with a monotone nonincreasing weight sequence is called briefly a *monotone owari*.

Property 1 *A monotone owari remains monotone after a move.*

Proof. It is a simple verification. □

Property 2 *If the weight sequence of an open owari has a last peak equal to the positive integer i then, after a derivation, the weight sequence has a last peak equal to $i - 1$.*

Proof. Let h be the tail of the open owari and let $w = [w_0, w_1, \ldots, w_{l-1}]$ be its weight sequence. After scooping the w_0 pebbles in h the sequence of weights of the successive holes, including h, becomes $w' = [0, w_1, w_2, \ldots, w_{l-1}]$, which has the same last peak i. By sowing the w_0 scooped pebbles into the holes following h we increase by 1 the terms of a subsequence of w' begining with w_1, and we add some terms equal to 1 at the end of w' if $w_0 > w_{l-1}$. This does not change the value of the last peak. The weight sequence after the derivation is obtained by deleting the null term from w', which decreases by 1 the value of the last peak. □

Corollary 1 *If an open owari has a weight sequence of length l, then it becomes monotone after $l - 2$ derivations.*

Proof. The last peak of that open owari is at most equal to $l - 1$. □

Let $F = [f_{ij} : 0 \le i < m, 0 \le j < n]$ be a $(0, 1)$-matrix. The column indices increase from left to right and the row indices from top to bottom. Matrix F is a *Ferrer matrix* if each of its rows and each of its columns is monotone nonincreasing. A Ferrer matrix is *reduced* if the topmost row and the leftmost column have no null entry. The transpose of a Ferrer matrix is a Ferrer matrix. The sequences $[\sum_{0 \le j < n} f_{ij} : 0 \le i < m]$ and $[\sum_{0 \le i < m} f_{ij} : 0 \le j < n]$ are the *row weight sequence* and the *column weight sequence* of F, respectively. The proofs of the following properties are easy.

Property 3 *Let F be a $(0, 1)$-matrix. If each column (row) of F is monotone nonincreasing and if the column (row) weight sequence of F is monotone nonincreasing, then F is a Ferrer matrix.*

Property 4 *If $w = [w_0, w_1, \ldots, w_{l-1}]$ is a monotone nonincreasing sequence of positive integers, then there is a unique reduced Ferrer matrix F with a column (row) weight sequence equal to w.*

Property 5 *If the row weight sequence of a Ferrer matrix satisfies Property* **AMG** *, then the column weight sequence also satisfies Property* **AMG** *.*

By using Property 4 we define the *Ferrer representation* of a Bulgarian solitaire (monotone owari) as the reduced Ferrer matrix with a row (column) weight sequence equal to the weight sequence of that Bulgarian solitaire (monotone owari).

Property 6 *The following transformation of a reduced Ferrer matrix F yields a reduced Ferrer matrix (see Figure 2 for an example):*

1. *Remove the leftmost column of F and transpose it into a row matrix R. Remove every null row from F.*
2. *If the numbers of columns of R and F are different make them equal by adding null columns at the right of the matrix with the smallest number of columns.*
3. *Insert R in F in such a way that the row weight function of F remains monotone nondecreasing.*

Proof. At the end of each step the rows are monotone nonincreasing and the row weight sequence remains monotone nonincreasing. Property 3 implies that the final matrix is a Ferrer matrix. □

$$
\begin{array}{ccccc}
\mathbf{1} & 1 & 1 & 1 & 1 \\
\mathbf{1} & 1 & 1 & 0 & 0 \\
\mathbf{1} & 1 & 0 & 0 & 0
\end{array}
\longrightarrow
\begin{array}{cccc}
\mathbf{1} & \mathbf{1} & \mathbf{1} & \\
 & & & \\
1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array}
\longrightarrow
\begin{array}{cccc}
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\
 & & & \\
1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array}
\longrightarrow
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array}
$$

Figure 2: Illustration of Property 6. The Ferrer matrix on the left corresponds to the Ferrer diagram in Figure 1. We first remove the leftmost column and transpose it into a row. The length of that row is adjusted to fit the width of the Ferrer matrix. Then it is inserted into that Ferrer matrix in order that the row weight sequence is monotone nonincreasing. The new Ferrer matrix represents the Bulgarian solitaire of Figure 1 after one move and the monotone owari after one derivation.

Property 7 *If F is the Ferrer representation of a Bulgarian solitaire (monotone owari) then, after applying the transformation of Property 6, F becomes the Ferrer representation of that Bulgarian solitaire (monotone owari) after one move (derivation).*

Proof. If F is the representation of a Bulgarian solitaire, then the matrix obtained after Step 1 represents this Bulgarian solitaire after one card has been removed from each pile and the new row inserted in Step 3 corresponds to the pile of cards that is added in order to complete the move.

If F is the representation of a monotone owari, then the matrix obtained after Step 1 represents this owari after we have scooped the pebbles in the tail and the new row inserted in Step 3 corresponds to sowing the pebbles into the holes following that former tail. □

Proof of Theorem 1. We prove that we get an augmented marching group after applying a sufficient number of derivations to an open owari. We first apply derivations in order to get a monotone owari, which is possible by Corollary 1. Then we construct the Ferrer representation F of that monotone owari and we consider a Bulgarian solitaire with a weight sequence equal to the row weight sequence of F. Let us perform a sufficient number of moves, say k, in order that the Bulgarian solitaire becomes periodic. We know by Property 7 that a move on the Bulgarian solitaire corresponds to a transformation of F, as described in Property 6. Therefore, after performing k transformations, we get a Ferrer matrix with a row weight sequence that satisfies Property **AMG** by Theorem 2. By Property 5 the column weight sequence of F also satisfies Property **AMG** , and so the monotone owari represented by F is an augmented marching group. Hence by invoking Property 7 again the k derivations applyed to the monotone owari lead to an augmented marching group. □

We could similarly derive Theorem 2 from Theorem 1.

3 A direct short proof of Theorem 1

Proof. We refer to [1] for verifying that an augmented marching group is periodic. To prove the converse we consider an owari O with a total number N of pebbles and we prove by induction on N that O is eventually transformed into an augmented marching group by applying repeated derivations. If $N = 1$ the result is obvious. Suppose $N > 1$.

Let O' be the open owari obtained from O by removing a pebble p, which we distinguish in the remainder. By induction O' becomes an augmented marching group, say after applying k moves. Apply these k moves to owari O by taking care, each time pebble p is scooped, to sow p as the last pebble. In that way the pebbles distinct from p are moved in the same way as in O'. Accordingly the pebbles distinct from p form an augmented marching group at the end of the k derivations. Let us possibly apply additional derivations until p falls into the tail and let us consider the owaris O and O' at this moment.

Since O' is a marching group, its weight sequence is of the form

$$[n + a_0, n - 1 + a_1, \ldots, 1 + a_{n-1}, \widehat{a_n}].$$

Since p is in the tail of O, the weight sequence of O is equal to

$$(1) \quad [n + a_0 + 1, n - 1 + a_1, \ldots, 1 + a_{n-1}, \widehat{a_n}].$$

If $a_0 = 0$ then O is an augmented marching group and the proof is completed. This is also the case if $a_i = 1$ for $0 \leq i \leq n$. It remains to consider the case where $a_0 = 1$ and $a_i = 0$ for some i such that $1 \leq i \leq n$. We complete the proof of that case by induction on the smallest value of i such that $a_i = 0$. By applying two derivations to O the weight sequence successively becomes

$$[n + a_1, n - 1 + a_2, \ldots, a_n + 1, a_0, 1]$$

and

$$[n + a_2, n - 1 + a_3, \ldots, a_0 + 1, a_1 + 1].$$

If $i = 1$, then $a_1 + 1 = 1$ and Property **AMG** is satisfied by the last weight sequence. Again the proof is completed. If $i > 1$ then, after applying n additional derivations to O, the weight sequence becomes

$$[n + a_1 + 1, n - 1 + a_2, \ldots, 1 + a_n, \widehat{a_0}].$$

If we let $b_0 = a_1, b_1 = a_2, \ldots, b_{n-1} = a_n, b_n = a_0$, then we are in the same situation as in (1), replacing a_i by b_i. But now the smallest value of i such that $b_i = 0$ has decreased by 1 and the induction on i is completed. □

4 Closed marching groups

Here we recall some results obtained by Bruhn [3]. A *closed owari* is made of finitely many pebbles distributed into finitely many holes placed along an oriented closed line. We assume the number of pebbles and the number of holes to be nonnull. In this paper we also distinguish a particular hole called the *active hole*. To *derive* a closed owari is to scoop the pebbles in the active hole and to sow them, one by one, into the subsequent holes. At the end of the sowing operation the active hole is replaced by the next hole. The number of pebbles in a hole is the *weight* of that hole. The number of holes is the *length* of the closed owari. The *weight sequence* $[w_0, w_1, \ldots, w_{l-1}]$ records the weights of the successive holes, starting with the weight w_0 of the active hole. A closed owari is *periodical* if we find back its weight sequence after a number $p > 0$ of derivations. Then the smallest value of p is the *period*.

Let l, q and r be three integers such that $0 < l$, $0 \leq q$ and $0 \leq r < l$ and let $M_{l,q,r} = [w_0, w_1, \ldots, w_{l-1}]$ be defined by $w_i = (l - i)q + \sup(0, r - i)$ for $i = 0, 1, \ldots, l - 1$. A *closed marching group* is a closed owari with weight sequence $M_{l,q,r}$, for some integers l, q and r. One verifies that a closed marching group has period 1.

Let l, q, r and $M_{l,q,r} = [w_0, w_1, \ldots, w_{l-1}]$ be defined as above, let $a = [a_0, a_1, \ldots, a_r]$ be a sequence of integers equal to 0 or 1 and let $M_{l,q,r,a} = [w_0 + a_0, w_1 + a_1, \ldots, w_r + a_r, w_{r+1}, \ldots, w_{l-1}]$. A closed owari with weight sequence $M_{l,q,r,a}$, for some l, q, r and a, is an *augmented closed marching group.* One verifies that an augmented closed marching group is a periodical closed owari.

Suppose $l > r$. The concatenation of the sequence $[r + a_0, r - 1 + a_1, \ldots, 1 + a_{r-1}, a_r]$ with a sequence of $l - r - 1$ null values is equal to the sequence $M_{l,0,r,a}$. Therefore every augmented marching group is a particular case of an augmented closed marching group. In that sense the following theorem, proved by Bruhn is a generalization of Theorem 1.

Theorem 3 *Every periodical closed owari is an augmented closed marching group.*

That theorem can be proved by adaptating the proof in Section 3.

Problem 1 *Let N and l be positive integers. Find an upper bound of the number of derivations to transform every closed owari with l holes and N pebbles into an augmented closed marching group.*

The similar problem for a Bulgarian solitaire with N cards (and by duality for a monotone owari with N pebbles) has been solved by G. Etienne [6], K. Igusa [8] and J. R. Griggs and Chih-Chang Ho [7]. A simple result says that if we let $N = 1 + 2 + \ldots + n + k$, with $0 \leq k < n$, then $n^2 - n$ is an upper bound. Better upper bounds and a conjecture are available in the last paper.

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