Microlocal High-Range-Resolution ISAR for Low Signal-to-Noise Environments

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Main idea

features of target $\longrightarrow$ features of data

$\uparrow$

microlocal analysis

Possible uses:

- Motion compensation and range alignment
- Discrimination of target from chaff
- Target ID directly from data
- Noise-free images

Ingredients

- Mathematical model for data
  - Kirchhoff (geometrical optics) model
  - Multiple scattering from point scatterers
  - Ducts
- Microlocal analysis
  - codification of high-frequency asymptotics
  - singularities (edges): location + orientation
  - used in modern geophysical processing
- Method for identifying features of (sampled, bandlimited) data

ISAR models: Kirchhoff scattering

$$u^s(t,\boldsymbol{x},\boldsymbol{x}) = \int_{\partial\Omega}\int \Big(g(t-t',\boldsymbol{x},\boldsymbol{y})\partial_{\boldsymbol{\nu}} u^s(t',\boldsymbol{y},\boldsymbol{x}) - u^s(t',\boldsymbol{y},\boldsymbol{x})\partial_{\boldsymbol{\nu}} g(t-t',\boldsymbol{x},\boldsymbol{y})\Big)\, dt'\, dS_{\boldsymbol{y}}$$

geometrical optics $\downarrow$ $u^s(t,\boldsymbol{y},\boldsymbol{x}) \approx V(\boldsymbol{y})\frac{\delta(t-|\boldsymbol{y}-\boldsymbol{x}|/c)}{4\pi|\boldsymbol{y}-\boldsymbol{x}|}$

$$u^s(t,\boldsymbol{x}) = \int_{\partial\Omega}\int i\omega \frac{\mathrm{e}^{-\mathrm{i}\omega(t-2|\boldsymbol{y}-\boldsymbol{x}|/c)}}{(8\pi^2|\boldsymbol{y}-\boldsymbol{x}|)^2} V(\boldsymbol{y}) 2\partial_{\boldsymbol{\nu}}|\boldsymbol{y}-\boldsymbol{x}|\,\mathrm{d}\omega\, dS_{\boldsymbol{y}}$$

target rotates, pulses starting at $t=\theta_n$, $\downarrow$ far-field approx.,

$\hat{\boldsymbol{R}}_n = -\mathcal{O}^{\mathrm{T}}(\theta_n)\hat{\boldsymbol{x}} \quad R = |\boldsymbol{x}|$

$$s_{\mathrm{sc}}(\boldsymbol{x},n,t) \approx \frac{2}{(8\pi^2 R)^2}\int_{\partial\Omega}\int Q_{\mathrm{K}}(\boldsymbol{z})(\mathrm{i}\omega)S_{\mathrm{inc}}(\omega)\,\mathrm{e}^{-\mathrm{i}\omega[t-\theta_n-(R+\hat{\boldsymbol{R}}_n\cdot\boldsymbol{z})/c]}\boldsymbol{\nu}\cdot\hat{\boldsymbol{R}}_n\,\mathrm{d}\omega\,\mathrm{d}S_{\boldsymbol{z}}\,.$$

correlation reception $\downarrow$ $r_n(\boldsymbol{z}) = R + \hat{\boldsymbol{R}}_n \cdot \boldsymbol{z}$

$$\eta_{\mathrm{K}}(\theta_n, t) = \frac{4\pi}{(8\pi^2 R)^2} \int_{\partial\Omega} \int (\mathrm{i}\omega) |S_{\mathrm{inc}}(\omega)|^2 \, \mathrm{e}^{-\mathrm{i}\omega[t - 2r_n(\boldsymbol{z})/c]} Q_{\mathrm{K}}(\boldsymbol{z}) \delta_{\partial\Omega}(\boldsymbol{z}) \boldsymbol{\nu} \cdot \hat{\boldsymbol{R}}_n \, \mathrm{d}\omega \mathrm{d}\boldsymbol{z} \,.$$

$\uparrow$ distance to target center

$\uparrow$ target scattering density

ISAR models: Scattering from Ducts

replace Q_{K} by

$$Q_{\mathrm{d}}(\omega, \theta_n, \boldsymbol{z}) = q_{\mathrm{M}}(\theta_n, \boldsymbol{z}) \sum_m \rho_m \, \mathrm{e}^{\mathrm{i}2L(\boldsymbol{z})c^{-1}\sqrt{\omega^2 - w_m^2}} \, .$$

coupling coefficient ↑ ↑ mode strength ↑ mode cutoff frequency

$$L(\boldsymbol{z}) = \begin{cases} L & \text{mouth} \\ 0 & \text{off mouth} \end{cases} \quad = \text{distance from the mouth to scatterer}$$

In time domain: $q_{\mathrm{d}}(t', \theta_n, \boldsymbol{z}) = \int Q_{\mathrm{d}}(\omega, \theta_n, \boldsymbol{z}) \, \mathrm{e}^{-\mathrm{i}\omega t'} \, \mathrm{d}\omega$

data is $\quad \eta_{\mathrm{d}}(\theta_n, t) = \iint K_{\mathrm{d}}(\theta_n, t, \boldsymbol{z}, t') q_{\mathrm{d}}(t', \theta_n, \boldsymbol{z}) \, \mathrm{d}t' \mathrm{d}\boldsymbol{z}$

where $K_{\mathrm{d}}(\theta_n, t, \boldsymbol{z}, t') = \dfrac{1}{(4\pi R)^2} \displaystyle\int (\mathrm{i}\omega) |S_{\mathrm{inc}}(\omega)|^2 \mathrm{e}^{-\mathrm{i}\omega[t_n - t' - 2r_n/c]} \, \mathrm{d}\omega \, .$

Wavefront Set of a Function f

Wavefront set $\mathrm{WF}(f)$ is a set in **phase space**

To determine whether a point $(\boldsymbol{x}, \boldsymbol{\xi})$ is in $\mathrm{WF}(f)$:

1. Localize around $\boldsymbol{x}$
2. Fourier transform $\boldsymbol{x} \to \boldsymbol{\xi}$
3. Examine decay in direction $\boldsymbol{\xi}$

slow decay $\leftrightarrow (\boldsymbol{x}, \boldsymbol{\xi}) \in \mathrm{WF}(f)$ fast decay $\leftrightarrow (\boldsymbol{x}, \boldsymbol{\xi}) \notin \mathrm{WF}(f)$

Example: a point scatterer. If $Q(\boldsymbol{x}) = \delta(\boldsymbol{x})$, then $\mathrm{WF}(Q) = \{(\mathbf{0}, \boldsymbol{\xi}) : \boldsymbol{\xi} \neq \mathbf{0}\}$.

Example: a specular flash. Suppose $Q(\boldsymbol{x}) = H(\boldsymbol{x} \cdot \boldsymbol{\nu})$, where H denotes the Heaviside function. Then $\mathrm{WF}(Q) = \{(\boldsymbol{x}, \alpha\boldsymbol{\nu}) : \boldsymbol{x} \cdot \boldsymbol{\nu} = 0, \alpha \neq 0\}$.

Key Facts about Wavefront Sets

- Wavefront sets can be arbitrary closed sets
- $\mathrm{WF}(fg) \subseteq (\mathrm{WF}(f) + \mathrm{WF}(g)) \cup \mathrm{WF}(f) \cup \mathrm{WF}(g)$ provided $\mathrm{WF}(f) + \mathrm{WF}(g) \equiv \{(\boldsymbol{x}, \boldsymbol{\xi}_f + \boldsymbol{\xi}_g)\}$ contains no points of the form $(\boldsymbol{x}, \mathbf{0})$
- Wavefront set of $f(\boldsymbol{x})$, considered as a function of both $\boldsymbol{x}$ and $\boldsymbol{y}$, is $\{(\boldsymbol{x}, \boldsymbol{y}); (\boldsymbol{\xi}, \mathbf{0})) : (\boldsymbol{x}, \boldsymbol{\xi}) \in \mathrm{WF}(f)\}$.
- If $K(\boldsymbol{x}) = \int \mathrm{e}^{\mathrm{i}\phi(\boldsymbol{\omega}, \boldsymbol{x})} a(\boldsymbol{x}, \boldsymbol{\omega})$, then $\mathrm{WF}(K) \subseteq \{(\boldsymbol{x}, \nabla_{\boldsymbol{x}}\phi) : \nabla_{\boldsymbol{\omega}}\phi(x) = \mathbf{0}\}$.
- A Fourier integral operator $f(\boldsymbol{x}) = \int K(\boldsymbol{x}, \boldsymbol{y}) g(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}$ maps $\mathrm{WF}(g)$ to $\mathrm{WF}(f)$ according to the (twisted) *canonical relation* $\Lambda' = \{[(\boldsymbol{x}; \boldsymbol{\xi}), (\boldsymbol{y}, \boldsymbol{\eta})] : (\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\xi}, -\boldsymbol{\eta}) \in \mathrm{WF}(K)\}$

Wavefront Set for the Kirchhoff Model

Canonical relation is

$$\Lambda' = \Big\{ (\theta_n, t : \sigma, \tau)(\boldsymbol{z}, \theta'; \boldsymbol{\zeta}, \sigma') : t_n - 2r_n(\boldsymbol{z})/c = 0, \qquad \theta_n = \theta'$$

$$\sigma = \partial_{\theta_n}\phi = -\tfrac{2\omega}{c}\frac{\partial \hat{\boldsymbol{R}}(\theta_n)}{\partial \theta_n} \cdot \boldsymbol{z} + \omega', \tau = \partial_t \phi = -\omega,$$

$$\boldsymbol{\zeta} = -\nabla_z \phi = \tfrac{2\omega}{c}\hat{\boldsymbol{R}}(\theta_n) \qquad \sigma' = -\partial_{\theta'}\phi = \omega' \Big\}$$

Wavefront set of $[\delta_\Omega(\boldsymbol{z}) Q_{\mathrm{K}}(\boldsymbol{z}) \boldsymbol{\nu}_{\boldsymbol{z}} \cdot \hat{\boldsymbol{R}}(\theta')]$ is

$\{(\boldsymbol{z}, \theta'; \boldsymbol{\zeta}, \sigma') : \ \boldsymbol{z} \in \partial\Omega, \ \theta' \text{ arbitrary}, \boldsymbol{\zeta} \propto \boldsymbol{\nu}_{\boldsymbol{z}}, \ \sigma' = 0\}$

$\Rightarrow$ Wavefront set of data is

$$\Big\{ (\theta_n, t; \sigma, \tau,) : \ t_n - 2r_n(\boldsymbol{z})/c = 0, \boldsymbol{z} \in \partial\Omega, \ \hat{\boldsymbol{R}}(\theta_n) \propto \boldsymbol{\nu}_{\boldsymbol{z}},$$

$$(\sigma, \tau) \propto \Big(-(2/c)\partial_{\theta_n}\hat{\boldsymbol{R}}(\theta_n) \cdot \boldsymbol{z}, 1 \Big) \Big\}.$$

Example: Single point scatterer at $\boldsymbol{z}^0$

Wavefront set is the curve

$$t_n - 2r_n(\boldsymbol{z}^0)/c = 0$$

with normal vector

$$(\sigma, \tau) \propto \left((2/c)\partial_{\theta_n}\hat{\boldsymbol{R}}(\theta_n) \cdot \boldsymbol{z}^0, 1\right)$$

How to extract wavefront set

(Assuming small-angle, bandlimited data from point scatterers)

Use CLEAN algorithm in data domain:

1. apply the generalized Radon-Hough transform to find the greatest-energy curve in the range-aspect data;
2. from this curve, use the microlocal theory to find the associated scattering center $\boldsymbol{z}$;
3. from the scattering center at $\boldsymbol{z}$, find the associated ambiguity function $\chi_{\boldsymbol{z}} = \int_{\omega_1}^{\omega_2} (\mathrm{i}\omega)|S_{\mathrm{inc}}(\omega)|^2 \, \mathrm{e}^{-\mathrm{i}\omega[t-2r_n(\boldsymbol{z})/c]} \, \mathrm{d}\omega$
4. subtract a (correct) multiple of this ambiguity function from the data;
5. return to step 1.

Results

SNR	x	y	mag.
True Values	−0.25	−0.25	1.00
	0.30	−0.10	0.60
	0.00	0.30	0.30
+30 dB	−0.252	−0.252	0.998
	0.302	−0.101	0.599
	0.013	0.300	0.294
−20 dB	−0.254	−0.256	0.987
	0.300	−0.097	0.621
	−0.015	0.290	0.291
−30 dB	−0.256	−0.245	1.005
	0.186	−0.202	0.079
	–	–	–

Wavefront set for duct data

Convert model into Fourier integral operator:

$$\eta_{\mathrm{d}}(\theta_n, t) = \iiint K_{\mathrm{d}}(\theta_n, \theta', t, \boldsymbol{z}, t') q_{\mathrm{d}}(t', \theta', \boldsymbol{z}) \, \mathrm{d}t' \mathrm{d}\theta' \, \mathrm{d}\boldsymbol{z} \,,$$

where

$$K_{\mathrm{d}}(\theta_n, \theta', t, \boldsymbol{z}, t') = \frac{1}{(4\pi R)^2} \iint (\mathrm{i}\omega)^2 |S_{\mathrm{inc}}(\omega)|^2 \mathrm{e}^{-\mathrm{i}\omega[t_n - t' - 2r_n/c]} \, \mathrm{e}^{\mathrm{i}\omega'(\theta_n - \theta')} \mathrm{d}\omega \mathrm{d}\omega' \,.$$

Wavefront set properties $\big\downarrow$

$$\mathrm{WF}(\eta_{\mathrm{d}}) \subseteq \Bigg\{ (\theta_n, t; \sigma, \tau) : \; t_n = \; (2r_n(\boldsymbol{z}) + 2L)/c, \quad \text{where } \boldsymbol{z} \in \; M,$$

$$A\left(\hat{\boldsymbol{R}}(\theta_n) \cdot \hat{\boldsymbol{N}}\right) \geq \; 0, \qquad (\sigma, \tau) \propto \left(-\frac{2\omega}{c} \frac{\partial \hat{\boldsymbol{R}}}{\partial \theta} \cdot \mathbf{z}, -1 \right) \Bigg\} \,.$$

Facts about wavefront set for duct data

- Curve in $\theta_n - t_n$ plane is associated with scattering centers lying within the duct/cavity at distance L from the mouth
- Critical curve is present in the data only at angles for which energy couples into the dispersive structure, and for times after which the wave has reached the scattering center within.

For more detail

- M. Cheney and B. Borden, "Microlocal structure of Inverse Synthetic Aperture Radar Data", *Inverse Problems* **19**, pp. 173–194, 2002.
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For background information

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