Synthetic Aperture Radar (SAR) and Microlocal Analysis

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data $\longrightarrow$ image $\longrightarrow$ edges $\longrightarrow$ information
texture
shapes

Synthetic Aperture Radar

- developed by the engineering community
- key technology is mathematics
- unknown in mathematical community
- close connections with tomography, integral geometry, and seismic inversion
- produces images that need processing and interpretation

History

1951	Carl Wiley, Goodyear Aircraft Corp.
mid-’50s	first operational systems DoD sponsorship: U. of Illinois, U. of Michigan, Goodyear Aircraft, General Electric, Philco, Varian
late ’60s	NASA sponsorship (unclassified!) first digital SAR processors
1978	SEASAT-A
1981	beginning of SIR (Shuttle Imaging Radar) series
1990s	satellites sent up by many countries SAR systems sent to Venus, Mars, Titan

Outline

1. model for the data

2. reconstruction algorithm and why it works (with microlocal tools)

3. resolution

4. sketch of the state of the art

Model for the data

- scalar wave (ignore polarization)

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right) u(t,x) = \text{source}$$

  has solution

$$u(t,x) = \frac{\delta(t - |x|/c)}{4\pi|x|} * \text{source}$$

- source = (incident wave at z) $\cdot\, V(z)$
  $V(z)$ = reflectivity function at z on ground
  (single scattering assumption)

- incident wave at z is proportional to

$$\int e^{i\omega(t-|z-x|/c)} a(z,x,\omega) d\omega.$$

  x = antenna location
  a contains:
    antenna beam pattern
    signal sent to antenna
    geometrical spreading factors

Data is of the form

$$D(t,x) = \int\int e^{i\omega(t-2|z-x|/c)} A(z,x,\omega) d\omega V(z) dz$$

From D, want to reconstruct V.

- seismic inversion problem with constant background
- $D(t,x)$ is known for x along a line or curve
- if $A(z,x,\omega) = 1$, want to reconstruct V from its integrals over spheres or circles
- D is an oscillatory integral, to which techniques of microlocal analysis apply
(Cliff Nolan & M.C.)
- character of problem depends on frequency content of A

High frequency case

- microwave frequencies, $\omega \sim 10^9$, $\lambda \sim$ cm
- use narrow beam, Fourier reconstruction
- high resolution, space-borne
- doesn't penetrate foliage

Low frequency case

- VHF: $\omega \sim 10^7$, $\lambda \sim$ 10m
- cannot form narrow beam
- low resolution, airborne
- penetrates foliage

Inversion scheme:

matched filter = backprojection = migration

$$I(y) = \int e^{-i\omega(t-2|x-y|/c)} B(x, y, \omega) d\omega D(t, x) dt dx$$

- Same phase as adjoint of map $F : V \to D$
  - similar to Fourier transform & Radon transform

- Integration over $x \longleftrightarrow$ synthetic aperture

- If we take $B = 1$:
  - "Backproject" the data over the sphere with radius $ct/2$ and center x.

  - Image at $y =$ sum of data from all spheres passing through y.

Analysis of Backprojection

$$I(y) = \int e^{-i\omega(t-2|x-y|/c)} B(x,y,\omega) D(t,x)\, d\omega dx dt$$

where B is to be determined below.

- Plug in expression for the data and do the t integration:

  $$I(y) \sim \int e^{i2\omega(|z-y|-|x-z|)/c} BA(\ldots) V(z)\, d\omega dz dx$$

- This is of the form $I(y) = \int K(y,z)V(z)dz$ where

  $$K(y,z) = \int e^{i2\omega(|z-y|-|x-z|)/c} BA(\ldots) d\omega dx$$

  is the **point spread function** or **generalized ambiguity function**

- Want K to look like a delta function

  $$\delta(y-z) = \int e^{i(y-z)\cdot\xi} d\xi$$

  .

$$K(y,z) = \int e^{i2\omega(|z-y|-|x-z|)/c} BA(\ldots) d\omega dx$$

G. Beylkin (JMP '85) approach:

1. Do Taylor expansion of exponent:

   $2\omega(|z-y|-|x-y|)/c = (y-z)\cdot\Xi(x,y,z,\omega)$

   near $y=z$, $\Xi(x,y,z,\omega) \approx (2\omega/c)\widehat{x-z}$

2. Make change of variables

$$(x,\omega) \to \xi = \Xi(x,y,z,\omega)$$

Then

$$K(y,z) = \int e^{i(y-z)\cdot\xi} BA(\ldots) \left|\frac{\partial(x,\omega)}{\partial\xi}\right| d\xi$$

Take $B = 1/(A\ |\partial(x,\omega)/\partial\xi|)$.
K is kernel of a **pseudodifferential operator** (has same phase as $\delta(y-z) = \int e^{i(y-z)\cdot\xi} d\xi$)
$\Rightarrow$ puts singularities (edges) in the correct locations, with correct orientation.

"Microlocal" study of singularities

Singularities in the scattering region correspond to boundaries between different materials. These singularities can have both a location and a direction.

(x^0, ξ) is **not** in the **wavefront set** of f if for some smooth cutoff function ϕ, the Fourier transform

$$\int f(x)\phi(x)e^{i\xi\cdot x}dx$$

decays rapidly in direction ξ.

To determine whether (x^0, ξ) is in the wavefront set:

1. localize around x^0
2. Fourier transform
3. look at decay in direction ξ

Example: a small "point" scatterer

$$V(x) = \delta(x) \propto \int e^{ix\cdot\xi} 1 d\xi \qquad i.e.,\ \mathcal{F}V = 1$$

does not decay in any direction.

$WF(\delta) = \{(0, \xi) : \text{all } \xi \neq 0\}$.

has singularities in all directions

Example: a wall

$$V(x) = \delta(x \cdot \widehat{\nu}) \propto \int e^{ix\cdot\widehat{\nu}\rho} d\rho$$

$$\Rightarrow \mathcal{F}V(\xi) = \int \delta(\xi - \widehat{\nu}\rho) d\rho$$

singularity in the direction $\widehat{\nu}$

The Fourier transform does not decay rapidly in direction $\widehat{\nu}$.

$$I(y) \sim \int K(y,z)\ V(z)dz$$

where

$$K(y,z) = \int e^{i(y-z)\cdot\xi}\chi(y,z,\xi)d\xi$$

is the kernel of a
pseudodifferential operator.

pseduodifferential operators have the
pseudolocal property:
$WF(Ku) \subseteq WF(u)$

- put singularities in the correct location

- do not change orientation of singularities

The pseudolocal property

Example: $V(z) = \delta(z \cdot \widehat{\nu}) \propto \int e^{iz\cdot\widehat{\nu}\rho} d\rho$
Then

$$\begin{aligned}\int K(y,z)\delta(z\cdot\widehat{\nu})dz &\propto \int K(y,z)e^{iz\cdot\rho\widehat{\nu}}d\rho dz \\ &= \int e^{i(y-z)\cdot\xi}\chi(y,z,\xi)e^{iz\cdot\rho\widehat{\nu}}\ d\xi dz d\rho,\end{aligned}$$

change variables $\xi \to \rho\tilde{\xi}$
large-ρ stationary phase reduction in z and $\tilde{\xi}$

$$\phi = \rho[(y-z)\cdot\tilde{\xi} + z\cdot\widehat{\nu}]$$

leading order contribution comes from:

$$d\phi/d\tilde{\xi} \Rightarrow z = y, \qquad d\phi/dz = 0 \Rightarrow \tilde{\xi} = \widehat{\nu}$$

(correct location) (correct orientation)

$$\int K(y,z)\delta(z\cdot\widehat{\nu})dz \propto \int \chi(y,y,\rho\widehat{\nu})e^{iy\cdot\rho\widehat{\nu}}d\rho + \text{(smoother)}$$

$\Rightarrow$ singularities are microlocally correct

Resolution

$$I(y) \sim \int K(y,z)\ V(z)dz$$

where

$$K(y,z) = \int e^{i(y-z)\cdot\xi}\chi(y,z,\xi)d\xi$$

Resolution via Fourier transforms

$$\int_{-b}^{b} e^{i\rho r} d\rho = \frac{2\sin br}{r}$$

corresponds to resolution $2\pi/b$.

$$K(y,z) = \int e^{i(y-z)\cdot\xi} \chi(y,z,\xi)\, d\xi$$

Recall

$$(x,\omega) \to \xi = \Xi(x,y,z,\omega)$$

where

$$\Xi(x,y,z) \approx (2\omega/c)\widehat{x-z}$$

$\Rightarrow$ resolution at y in direction $\hat{\nu}$ is determined by the size of the ρ-interval for which $\chi(y,z,\rho\hat{\nu})$ is nonzero.

$$\rho\hat{\nu} = \xi \approx \frac{2\omega}{c}\widehat{x-z}$$

Resolution determined by bandwidth and extent of survey

Example: Along-track resolution in high frequency case

$$K(y,z) = \int e^{i(y-z)\cdot\xi} \chi(y,z,\xi) d\xi \qquad \xi \approx \frac{2\omega}{c} \widehat{x-z}$$

If flight track is $x(s) = (0, s, H)$ and $y - z = (0, y_2 - z_2, 0)$, need only

$\xi_2 \approx \frac{2\omega}{c} \frac{x_2 - z_2}{R} = \frac{\omega}{cR}\, 2(s - z_2)$

But
$2 \max |s - z_2| =$ width of antenna footprint
$= \frac{2\lambda}{L} R = \frac{4\pi c}{\omega L} R$
$=$ **effective length of synthetic aperture**

So $\max |\xi_2| \approx \frac{\omega}{cR}\ \frac{4\pi c R}{\omega L} = \frac{4\pi}{L}$

So resolution in along-track direction is

$$\frac{2\pi}{4\pi / L} = \frac{L}{2}$$

Along-track resolution is $L/2$.

This is ...

- independent of range!
- independent of λ!
- better for small antennas!

These are all explained by noting that when a point z stays in the beam longer, the effective aperture for that point is larger.

In range direction, want broad frequency band $\Rightarrow$ get largest coverage in ξ.

Microlocal viewpoint

Advantages

- Microlocal tools are built to study singularities – want edges
- Provides theory for backprojection algorithm
- Shows how to correct for antenna beam pattern, signal waveform, geometrical spreading, variable topography, etc.
- Formula reduces to known exact one in idealized case
- Analysis of change of variables gives information about artifacts

Disadvantages

- Applies to linearized problem
- Asymptotic high-frequency theory

What can be done with radar

- Can correct for antenna motion
- **interferometric** SAR (two flight passes or two antennas)
  - topographic information
  - changes (glacier flow, volcano movement)
- **polarimetric** SAR (uses full Maxwell's equations)
- detection of moving targets
  MTI = Moving Target Indicator
- SAR in **wave mode**: Use subaperture information to get ocean motion

Current research and open problems

- Foliage-penetrating (FOPEN) SAR
  - low frequencies $\Rightarrow$ poor directivity $\Rightarrow$ tomographic image formation from integrals over circles or spheres
  - want bare earth topography
  - want to find tanks under trees
  - want to estimate forest biomass, trunk volume, tree health
- Ground-penetrating radar (GPR)
  - land mines
  - unexploded ordinance (UXO)
- Ultra-wideband (UWB) SAR

- Improved modeling
  - dispersion (FOPEN)
  - multiple scattering
    - rough terrain, wave-guiding structures
    - wave propagation in random media, clutter

- Image interpretation
  - thickness of ice, species of trees, health of vegetation, ...
  - Automatic Target Recognition (ATR) (school bus or tank?)
  - sensor fusion, use of multiple frequency bands

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