Determination of Weight Vector (cardinal ratio scale) from Pairwise Comparisons

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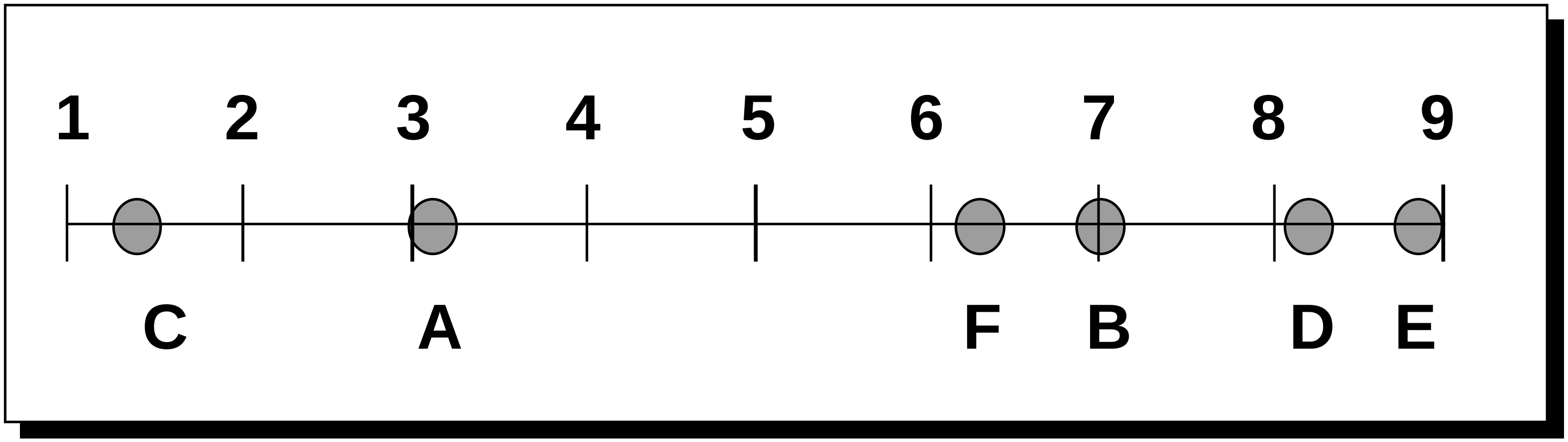
Weight Determination via Saaty's Method

- Motivation
  - People are inconsistent in eliciting weights, i.e., in providing a *cardinal* scale to evaluate objects according to some subjective preference criteria



$$[\, W_{A}, W_{B}, W_{C}, W_{D}, W_{E}, W_{F} \,] = [\, 3.12, 7.01, 1.3, 8.1, 8.8, 6.22 \,]$$

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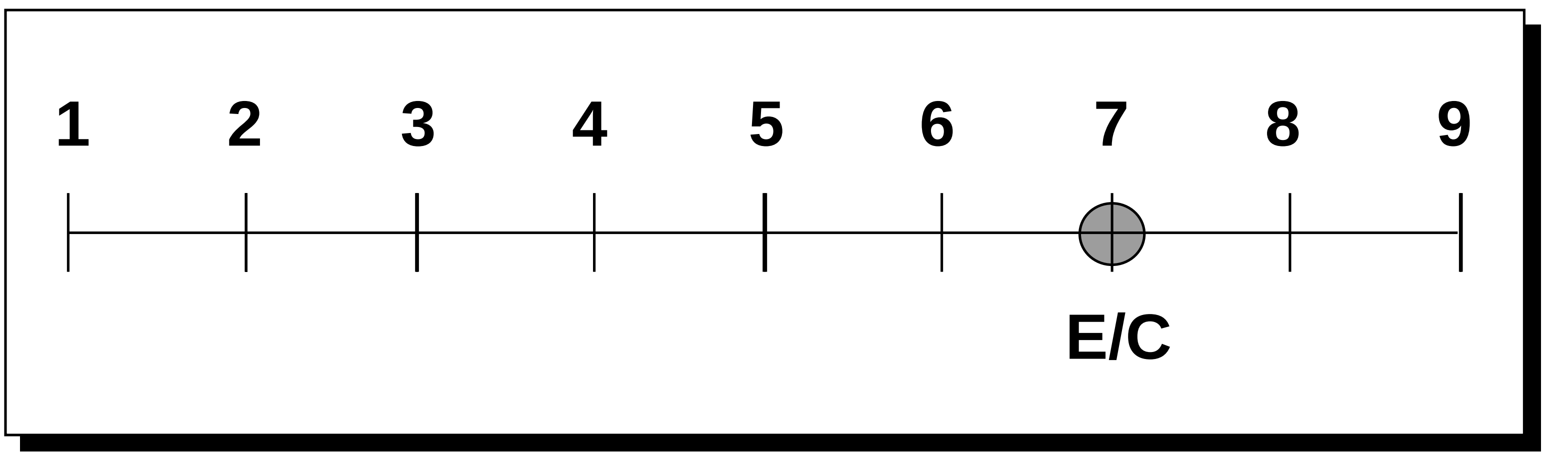
Weight Determination via Saaty's Method

- Motivation (cont.)
  - They are better at comparing *pairs* of objects:

"E is much better than C"



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Nine Point Preference Scale

- A nine point scale is provided to quantify pairwise importance (or preference):

1	Equal Important	- Indifferent
3	Weak Importance	- Slightly better
5	Strong Importance	- Better
7	Very Strong Importance	- Much Better
9	Absolute Importance	- Definitely MuchBetter

- Intermediate values used to interpolate between adjacent scale values

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Goal

- Goal:
  - Generate an *n*-dimensional Cardinal Ratio Scale

    $$\vec{W}$$

    from a *nxn* Matrix [A] obtained from

    $$\binom{n}{2} = \frac{n\,(n+1)}{2}$$

    pairwise comparisons

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Matrix [A]: Structure

- Given *n* objects, let [A] = [$a_{j,k}$] be a *nxn* reflexive matrix obtained by evaluating $\binom{n}{2}$ *pairwise comparisons*, such that:

a_{jk} takes values in {1,…, 9}

$a_{jj} = 1$ for all j=1,…,n [reflexive]

$a_{jk} = 1/a_{kj}$ [transpose is inverse]

$$a_{jk} = w_j / w_k$$

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Assumptions and Structure of A

- Constructive Assumption

$$a_{ij} > 0$$

(1) *[positive entries]*

- Consistency Assumption

$$a_{ij} \times a_{jk} = a_{ik}$$

(2) *[transitivity]*

- Then

$$a_{ii} = 1 \quad \forall i$$

[reflexive]

$$a_{ij} = \frac{1}{a_{ji}}$$

[transpose is inverse]

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Proof

Proof

Setting *k=1* in the Consistency Assumpt. (2):

$$a_{ij} \times a_{ji} = a_{ii} \qquad (3)$$

Setting *j=i, k=j* in the Consistency Assumpt. (2):

$$a_{ii} \times a_{ij} = a_{ij} \qquad (4)$$

Since from the Constructive Assumption $a_{ij} \neq 0$

then Equation (4) implies $a_{ii} = 1 \quad \forall i$

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Proof (cont.)

by using this reflexive property in (3):

$$a_{ij} \times a_{ji} = 1$$

$$a_{ij} = \frac{1}{a_{ji}}$$

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Weights Determination

- Let

$$\vec{W} = [w_1, \ldots w_n]^T$$

be the Weight Vector that we try to determine

- We obtain the matrix [A] by performing $\binom{n}{2}$ pairwise comparison using the nine point scale
- We multiply $[A] \times \vec{W}$

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Eigenvalue Equation for Ideal Matrix [A]

Clearly $$[A] \times \vec{W} = n\,\vec{W}$$

since

$$\begin{bmatrix} 1 & \frac{w_1}{w_2} & & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & 1 & & \frac{w_2}{w_n} \\ & & & \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \cdots \\ w_n \end{bmatrix} = \begin{bmatrix} n\,w_1 \\ n\,w_2 \\ \cdots \\ n\,w_n \end{bmatrix}$$

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Eigeinvalue Equation (cont)

- Therefore, given the equation

$$[A]\times\vec{W} = n\,\vec{W}$$

we note that *n* is an eigenvalue of matrix A

- Furthermore, since the dimension of A is n:

$$trace(A) = \sum_{i=1}^{n} \lambda_i = n$$

- Since the entries of A are non-negatives, all of A's eigenvalues are non-negative

$$\forall\, i, \lambda_i \geq 0$$

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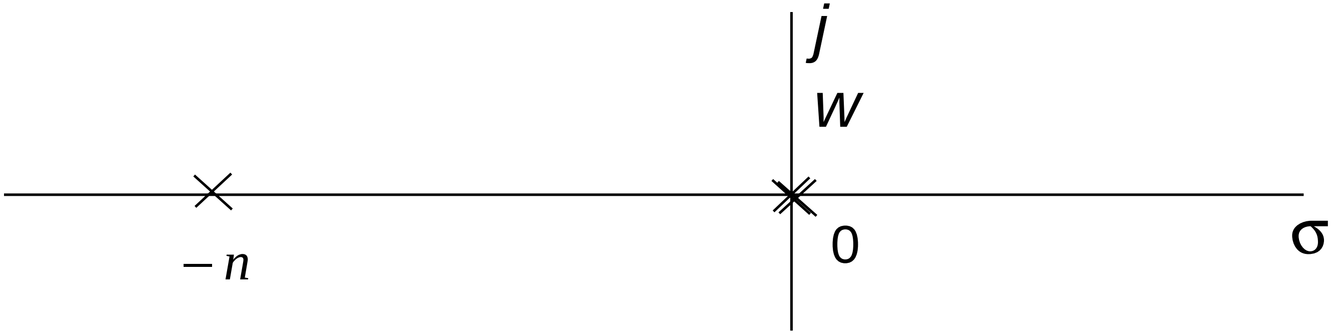
Eigeinvalue Equation (cont)

- As a result:

$$\lambda_{\max} = \lambda_1 = n$$

$$\lambda_i = 0 \quad \text{for } i = 2, \ldots, n$$



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Eigeinvalue Equation for Real Matrix [A’]

- This means that ideal matrix [A] is of rank 1:
  - only one row is independent, and using the consistency assumption we can recreate all other rows from the first one.

$$a_{ij} \times a_{jk} = a_{ik}$$

- However, as we elicit the real matrix [A’], the person providing the pairwise comparisons may not be consistent and transitivity may not hold:

$$a_{ij} \times a_{jk} \neq a_{ik}$$

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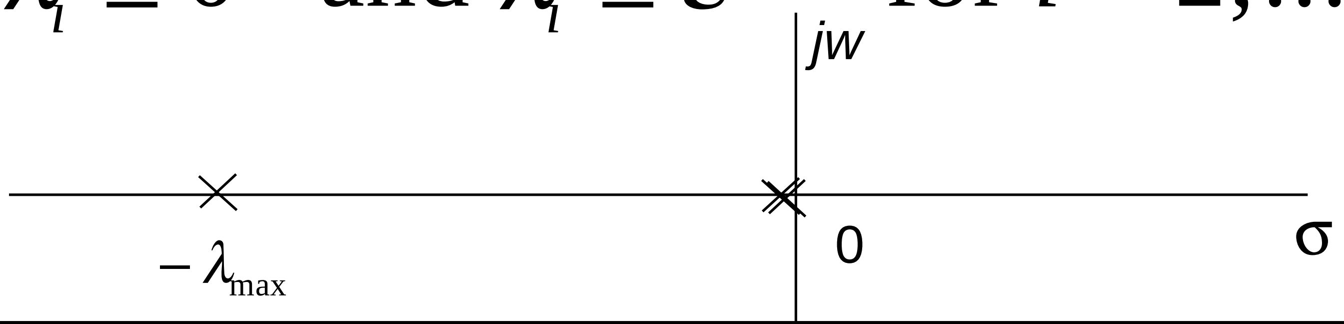
Eigeinvalue Perturbation for [A’]

- If there are small changes between the entries of [A’] and [A], then we can treat the case as an eigenvalue perturbation:
  - The maximum eigenvalue of [A’] will be around *n*
  - The other eigenvalues will be positive, around 0

$$\lambda_{\max} = \lambda_1 \approx n$$

$$\lambda_i \geq 0 \quad \text{and } \lambda_i \leq \varepsilon \qquad \text{for } i = 2, \ldots, n$$



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Solution

- Therefore:

$$[A'] \times \vec{W} = \lambda_{\max} \vec{W}$$

- The Weight Vector $\vec{W}$ is the Eigenvector of [A’] corresponding to its maximum eigenvalue $\lambda_{\max}$
- The distance between the maximum eigenvalue and n is a measure of user’s inconsistency: $\frac{|\lambda_{\max} - n|}{n}$

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Power Method

- The power method can be used to quickly obtain $\vec{W}_1$, the Maximum Eigenvector of [A']:

$$[A'] \times \vec{W}_1 = \lambda_{\max} \vec{W}_1$$

- Applicability Assumption:
  - A' has n linearly independent eigenvectors $\vec{W}_i$ and a *unique* eigenvalue of maximum magnitude

$$|\lambda_1| > |\lambda_2| \geq \ \ldots \geq |\lambda_n|$$

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Power Method

1) Randomly generate an *n*-dimensional vector:

$$\vec{Z}_0 = \gamma_1 \vec{W}_1 + \ldots + \gamma_n \vec{W}_n$$

2) Define

$$\vec{Z}_k = \frac{[A']^k \vec{Z}_0}{\lambda_1^k} = \frac{\begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}^k \vec{Z}_0}{\lambda_1^k}$$

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Power Method

3) Then:

$$\vec{Z}_k = \gamma_1 \vec{W}_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^k \gamma_2 \vec{W}_2 + \ldots + \left(\frac{\lambda_n}{\lambda_1}\right)^k \gamma_n \vec{W}_n$$

4) Since

$$\left|\frac{\lambda_i}{\lambda_1}\right| < 1 \text{ for } i = 2, \ldots, n$$

5) Then

$$\lim_{k \to \infty} \left(\frac{\lambda_i}{\lambda_1}\right)^k = 0$$

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Power Method

6) So: $$\lim_{k \to \infty} \vec{Z}_k = \gamma_1 \vec{W}_1$$

provided that $\gamma_1 \neq 0$

7) For computational reasons, we define Z_k recursively, using a scale factor σ_k instead of λ_1 :

$$\vec{Z}_{k+1} = \frac{[A'] \ \vec{Z}_k}{\sigma_k}$$

where: $$\sigma_k = \max_{k=1}^{n} Z_k$$

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Example - see Excel File

Real

$$[A'] = \begin{bmatrix} 1 & 4 & 3 & 7 \\ 1/4 & 1 & 1 & 2 \\ 1/3 & 1 & 1 & 2 \\ 1/7 & 1/2 & 1/2 & 1 \end{bmatrix} \qquad \vec{W}_1 = \begin{bmatrix} 1 \\ 0.2775 \\ 0.2983 \\ 0.1432 \end{bmatrix}$$

Ideal

$$[A] = \begin{bmatrix} 1 & 4 & 3 & 7 \\ 1/4 & 1 & 3/4 & 7/4 \\ 1/3 & 4/3 & 1 & 7/3 \\ 1/7 & 4/7 & 3/7 & 1 \end{bmatrix} \qquad \vec{W}_1 = \begin{bmatrix} 1 \\ 0.2500 \\ 0.3333 \\ 0.1428 \end{bmatrix}$$

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Final Normalization

- Finally the eigenvector

$$\vec{W}_1 = \begin{bmatrix} 1 \\ 0.2775 \\ 0.2983 \\ 0.1432 \end{bmatrix}$$

should be normalized so that the sum of its elements is equal to *n*

$$Norm(\vec{W}_1) = \begin{bmatrix} 2.326 \\ 0.645 \\ 0.694 \\ 0.333 \end{bmatrix}$$

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