Fuzzy Sets & Expert Systems in Computer Eng. (5):

Linguistic Variables

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Outline

- **Linguistic variable**
- **Language**
- **Grammars**
- **Notation and Operations for Sampled Membership Functions**
- **Notation and Operations for Parametrized Membership Functions**
  - Possible Problems with Convexity and Normality
- **Bug Report**
- **Defuzzification Methods**

Linguistic Variables

A numerical variables takes numerical values:

Age = 65

A linguistic variables takes linguistic values:

Age is old

A linguistic values is a fuzzy set.

It has a LABEL and a MEANING

LABEL: Symbol, Sentence in a Language

MEANING: Fuzzy Subset of a Universe of Discourse

Linguistic Variables

All linguistic values form a term set:

T(age) = {young, not young, very young, ...
middle aged, not middle aged, ...
old, not old, very old, more or less old, ...
not very young and not very old, ...}

Linguistic Variable (Formal Definition)

A Linguistic Variable is characterized by the five-tuple:

$$(\sigma , T(\sigma), U, G, A)$$

where:

σ	**is the name of the variable**
$T(\sigma)$	**is the Term Set of σ**
U	**is the Universe of Discourse associated with the base variable “u”**
G	**is the Syntactic Rule (grammar) for generating the labels “t” in the termset**
A	**is the Semantic Rule for associating to each label t, element of $T(\sigma)$, its meaning A(t) - algorithmic procedure**

Language (Zadeh 71)

A language can be seen as a fuzzy relation from a set of labels T to a Universe of Discourse U, which assigns a grade of membership $\mu_L(t, u)$ to each pair (t, u) ε TxU

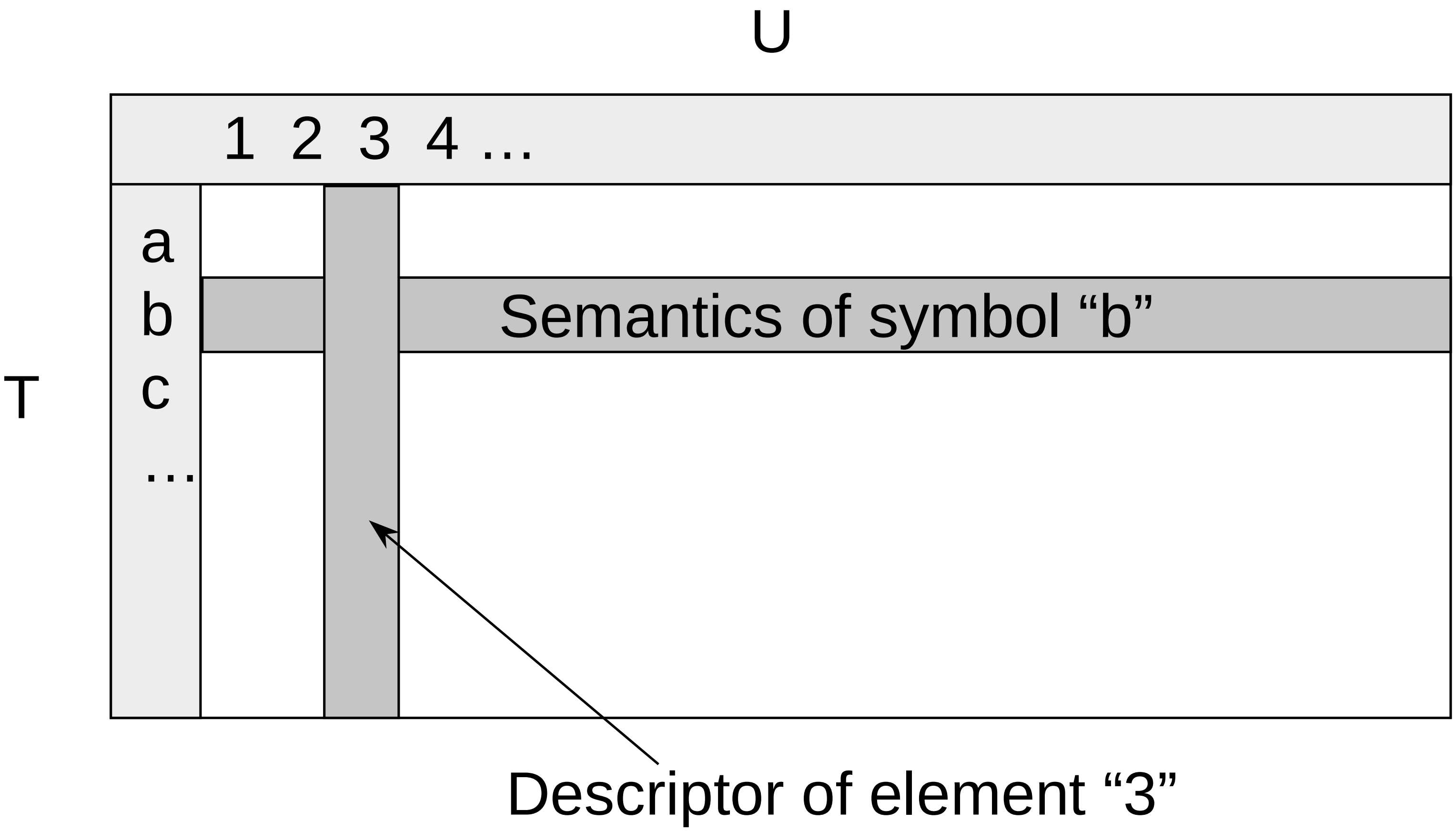
If we fix the term "t" say t_i, then $\mu_L(t_i, u)$ determines the fuzzy subset $A(t_i)$ whose membership function is $\mu_{A(ti)}(u) = \mu_L(t_i, u)$

$\mu_{A(ti)}(u)$ is the MEANING of the linguistic value whose LABEL is t_i

If we fix the term u, say u_j, then $\mu_L(t, u_j)$ is the DESCRIPTOR of u_j

Language (Zadeh 71)

A language is a fuzzy mapping UxT =>[0,1]



Linguistic Terms (Termset)

		Universe of Discourse								
Label	**Symbol**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Very Low	Ver_L	1	0.64	0.09	0.01	0	0	0	0	0
Low	L	1	0.80	0.30	0.10	0	0	0	0	0
More or less Low	Mor_L	1	0.89	0.55	0.32	0	0	0	0	0
Above Low	> L	0	0.20	0.70	0.90	1	1	1	1	1
Below Medium	< M	1	1.00	0.90	0.25	0	0	0	0	0
Above Low and Below Medium		0	0.20	0.70	0.25	0	0	0	0	0
Between Medium and H	L-M	0	0.29	1	0.36	0	0	0	0	0
Medium	M	0	0	0.10	0.75	1	0.75	0.10	0	0
Above Medium	>M	0	0	0	0	0	0.25	0.90	1	1
Below High	< H	1	1	1	1	1	0.90	0.70	0.20	0
Above Medium and Below High		0	0	0	0	0	0.25	0.70	0.20	0
Between Medium and H	M-H	0	0	0	0	0	0.36	1	0.29	0
More or less High	Mor_H	0	0	0	0	0	0.32	0.55	0.89	1
High	H	0	0	0	0	0	0.10	0.30	0.80	1
Very High	Ver_H	0	0	0	0	0	0.01	0.09	0.64	1

Generative Grammars

A grammar is a set of rules and procedures which produces a structurally correct language.

A generative grammar G is defined by the four-tuple:

$$G = < V_H , V_E ,P, S>$$

where:

V_H	**is a finite, non-empty set of Help or *non-terminal* vocabulary of G**
V_E	**is a finite, non-empty set of End or *terminal* vocabulary of G**
P	**is a finite, non-empty set of ordered pairs (l,r), where P is in V*xV*. These pairs are called production rules (l -> r)**
S	**is the starting symbol (an element of V_H)**

Generative Grammars and Languages

A Language L(G) is the set of strings made only with terminal symbols, that can be generated from the starting symbol S

$$L(G) = \{w | S \xrightarrow{*} w, w \in V_E^*\}$$

Generative Grammars and Languages (Ex 1)

$$G = \left\langle \{S\}, \{a, b\}, \{S \rightarrow aS, S \rightarrow b\}, S \right\rangle$$

L(G) = b, ab, aab, aaab, …

= { $a^n b$ | n ≥ 0 }

Types of Generative Grammars

Chomsky defines four types of grammars, depending on the restrictions imposed on P:

0 unrestricted

1 Context-Sensitive

$$n_1 A n_2 \rightarrow n_1 w n_2$$

$$\text{where } n_1, n_2 \in (V_H \cup V_E)^*, A \in V_H$$

$$\text{and } w \in (V_H \cup V_E)^+ \text{ except } S \rightarrow \varepsilon$$

2 Context-Free

$$A \rightarrow w \text{ where } A \in V_H \text{ and } w \in (V_H \cup V_E)^*$$

3 Regular - each production is of the type

$$A \rightarrow a \text{ or } A \rightarrow aB \text{ where } A, B \in V_H \text{ and } a \in V_E$$

Generative Grammars (Ex 2)

$$S \to NP\ \ (AUX)\ \ VP |\, AUX\ \ NP\ \ VP$$

$$NP \to (ART)\ \ (ADJ*)\ \ N\ \ (PP*)$$

$$VP \to V\ \ (NP)\ (PP*)$$

$$PP \to PREP\ \ NP$$

$$ART \to the | a | an$$

$$N \to John | Mary | book | \ldots$$

...

See Augmented Finite State Transition Network

Notation for Sampled Membership Functions

Let $\mathbf{X}$ be the N-dimensional vector $[X(u_1), ..., X(u_n)]$

All the vector operations used in the definitions are to be computed element-wise, so that the notation f(X) is equivalent to applying f(.) to each element $X(u_i)$, e.g.:

$\mathbf{X}^2 = [X(u_1)^2, ..., X(u_n)^2]$

$1- \mathbf{X} = [1-X(u_1), ..., 1-X(u_n)]$

$\text{Max}\ \{\mathbf{X}, \mathbf{Y}\} = [\max\{X(u_1), Y(u_1)\} ..., \max\{X(u_n), Y(u_n)\}]$

Semantics for Hedges and Connectors Defined Sampled Membership Functions

X *and* Y ⇨ min {X, Y}
X *or* Y ⇨ max {X, Y}
not X ⇨ 1 - X
very X ⇨ CONTR(X) = X^2
extremely X ⇨ CONTR(CONTR(X)) = X^4
more_or_less X ⇨ DIL(X) = $X^{0.5}$
indeed X ⇨ INT(X)
hardly X ⇨ FUZ(X)
younger_than X ⇨ SM(X)
older_than X ⇨ GR(X)
in_between (X)(Y) ⇨ NORM{GR(X) *and* SM(Y)}
⇨ NORM{*not* SMEQ(X)) *and* *not* GREQ(Y))}
from_to (X)(Y) ⇨ GREQ(X) *and* SMEQ(Y)

Semantics for Hedges and Connectors Defined Sampled Membership Functions

where:

NORM(X) $= X / \text{Max}_i\, X(u_i)$

INT(X) $= 2X^2$ when $X < 0.5$

$= 1-2(1-X)^2$ when $X > 0.5$

FUZ(X) $= 0.5 - 2(0.5-X)^2$ when $X < 0.5$

$= 0.5 + 2(X-0.5)^2$ when $X > 0.5$

SM(X) = *not* GREQ(X)

GR(X) = *not* SMEQ(X)

GREQ(X) = X for $u_i < u^*$

= 1 for $u_i > u^*$

SMEQ(X) = 1 for $u_i < u^*$

= X for $u_i > u^*$

Notes:

- $u^* = \text{Min}\,\{u_i \mid X(u_i) = 1\}$ (leftmost value of U with membership =1)
- If needed, normalize X(u) before using SMEQ(X) or GREQ (X)

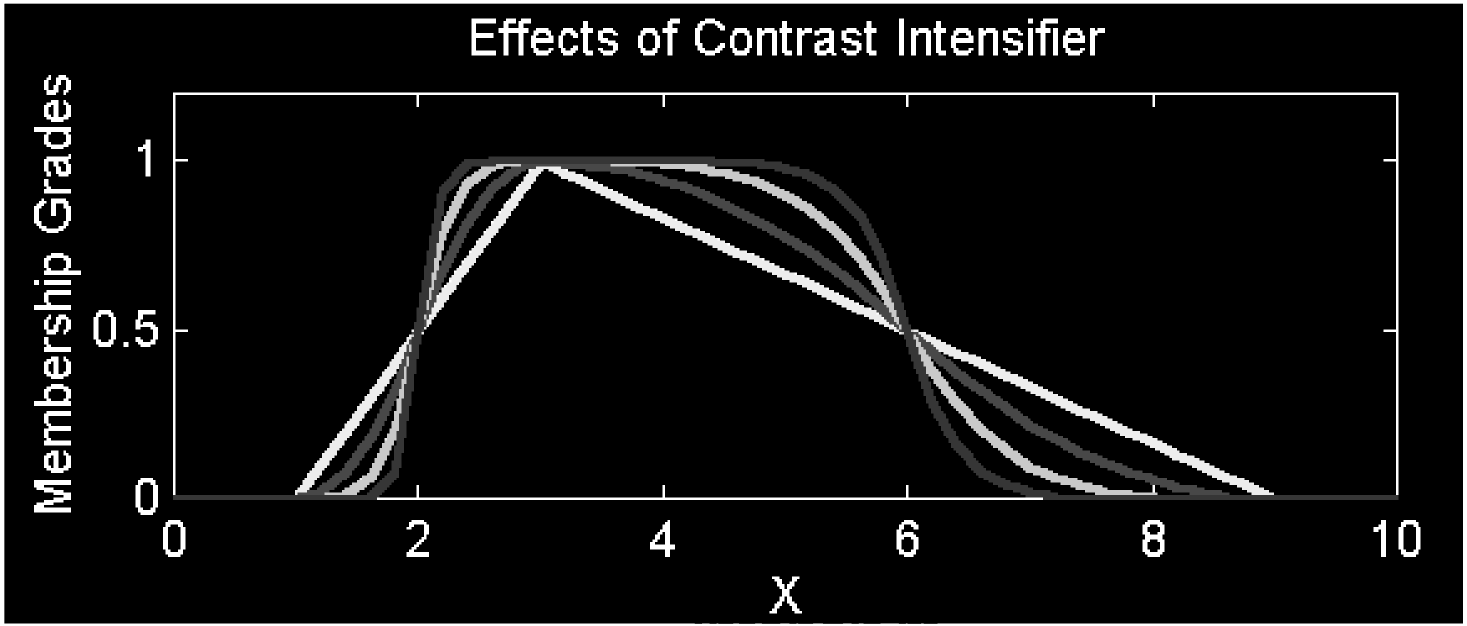
Operations on Linguistic Values

Concentration: $CON(X) = X^2$

Dilation: $DIL(X) = X^{0.5}$

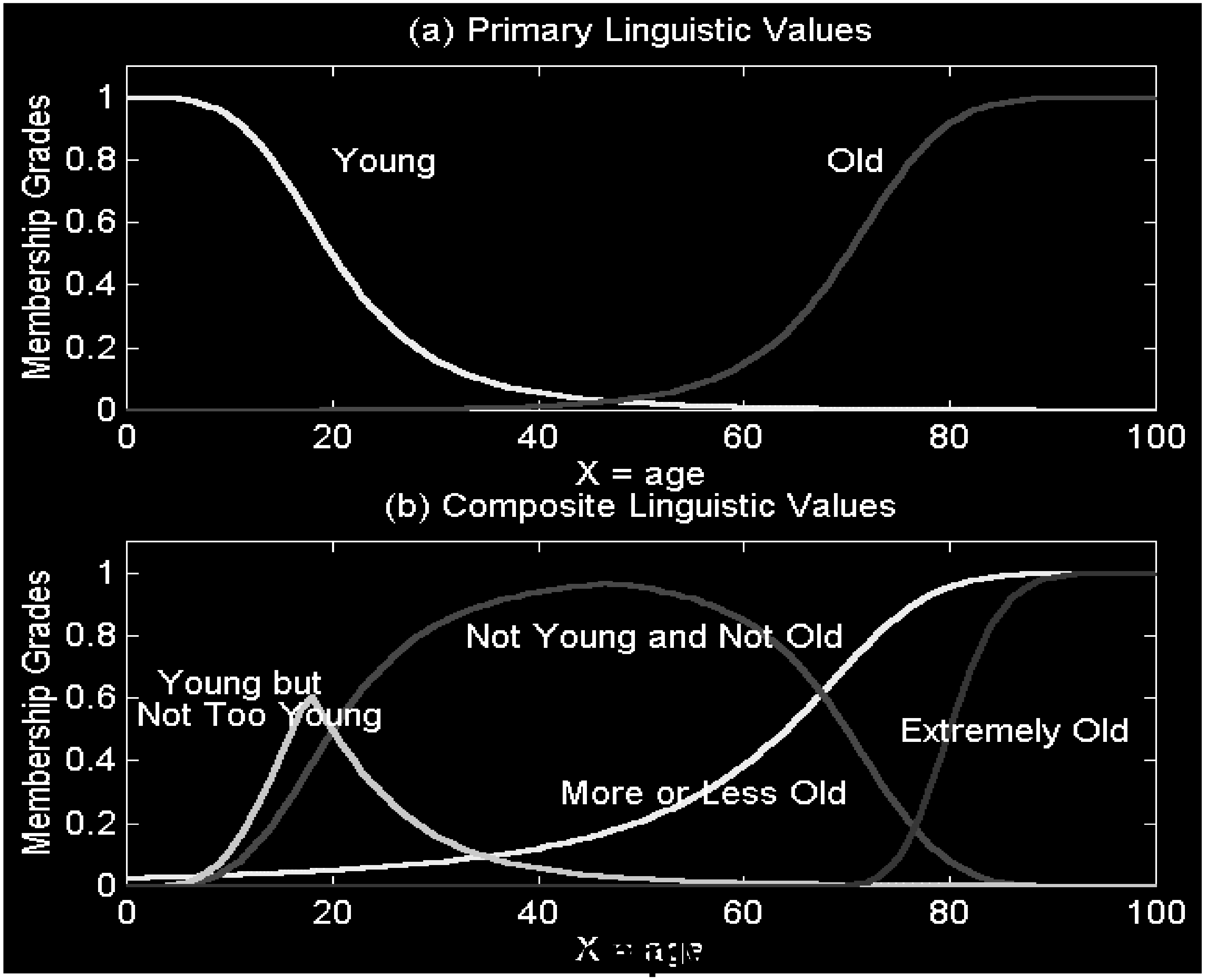
Contrast intensification:

$$INT(X) = \begin{cases} 2X^2, & 0 \le \mu_X(x) \le 0.5 \\ 1-2(1-X)^2, & 0.5 \le \mu_X(x) \le 1 \end{cases}$$



intensif.m

Linguistic Values (Terms)



Notation for Parametrized Membership Functions (for Optional steps)

- Each *primary term* is now represented by a 4-parameter list.
- Let X = (a, b, α, β) and Y = (c, d, γ, δ).
- Let the universe of discourse be the interval [w,z].

Note:

- This parametric representation is valid only for *convex*, *normal* Fuzzy Numbers.

- For this optional step, disregard the terms derivable from non-terminal U, because neither *no_age* nor *unknown_age* are normal fuzzy numbers.

- Therefore, they cannot be represented by the four parameter representation.

Operations (*and*) for Parametrized Membership Functions

X and Y & = (p, q, L_a, R_a)

$$p = \max(a, c)$$

$$q = \min(b, d)$$

$$\begin{array}{lll}
L_a = (c + \alpha) - \min(a, c) & \text{if} & (a - \alpha) > (c - \gamma) \\
\quad = (a + \gamma) - \min(a, c) & \text{if} & (a - \alpha) < (c - \gamma) \\
\quad = \alpha & \text{if} & (a - \alpha) = (c - \gamma),\ a > c \\
\quad = \gamma & \text{if} & (a - \alpha) = (c - \gamma),\ a < c \\
\quad = \alpha = \gamma & \text{if} & (a - \alpha) = (c - \gamma),\ a = c
\end{array}$$

$$\begin{array}{lll}
R_a = (d + \delta) - \min(b, d) & \text{if} & (b + \beta) > (d - \delta) \\
\quad = (b + \beta) - \min(b, d) & \text{if} & (b + \beta) < (d - \delta) \\
\quad = \delta & \text{if} & (b + \beta) = (d - \delta),\ b > d \\
\quad = \beta & \text{if} & (b + \beta) = (d - \delta),\ b < d \\
\quad = \beta = \delta & \text{if} & (b + \beta) = (d - \delta),\ b = d
\end{array}$$

Possible Problems with the ***and*** Operator and Normality

If max (a, c) ≤ min(b, d), the result is still a *normal* set.

Otherwise, normalize the result by finding the intersection of the two slopes, and scaling the membership value of that point b (making it equal to one):

$$\mathbf{X\ \underline{\mathit{and}}\ Y = NORM\ (p, q, L_a, R_a) = (x, x, L_n, R_n)}$$

where

$$x = (q\,R_{\mathbf{a}} + p\,L_{\mathbf{a}})/(L_{\mathbf{a}} + R_{\mathbf{a}})$$

$$L_n = x - (p - L_{\mathbf{a}})$$

$$R_n = (q + R_{\mathbf{a}}) - x$$

Operations (*or*) for Parametrized Membership Functions

X *or* Y = (r, s, L_o, R_o)

$$r = \min(a, c)$$

$$s = \max(b, d)$$

$$\begin{aligned}
L_o &= (a + \gamma) - \max(a, c) && \text{if } (a - \alpha) > (c - \gamma) \\
&= (c + \alpha) - \max(a, c) && \text{if } (a - \alpha) < (c - \gamma) \\
&= \gamma && \text{if } (a - \alpha) = (c - \gamma),\ a > c \\
&= \alpha && \text{if } (a - \alpha) = (c - \gamma),\ a < c \\
&= \alpha = \gamma && \text{if } (a - \alpha) = (c - \gamma),\ a = c
\end{aligned}$$

$$\begin{aligned}
R_o &= (b + \beta) - \max(b, d) && \text{if } (b + \beta) > (d - \delta) \\
&= (d + \delta) - \max(b, d) && \text{if } (b + \beta) < (d - \delta) \\
&= \delta && \text{if } (b + \beta) = (d - \delta),\ b > d \\
&= \beta && \text{if } (b + \beta) = (d - \delta),\ b < d \\
&= \beta = \delta && \text{if } (b + \beta) = (d - \delta),\ b = d
\end{aligned}$$

Possible Problems with the *<u>or</u>* Operator and Convexity

If $\max(a, c) \leq \min(b, d)$ the result is still a *convex* set.

Otherwise, maintain both fuzzy sets as elements of a data structure called UNION-LIST, i.e.,

X *<u>or</u>* Y = UNION-LIST(X,Y).

Any subsequent operation on the result of X *<u>or</u>* Y can be distributed to each element in UNION-LIST.

If $\max(a, c) \leq \min(b, d)$, the result is still a *normal* set.

Remaining Definitions

not X ⇨ UNION-LIST(SM(X), GR(X))

very X ⇨ $CONTR(X) = (a + \alpha/5, b - \beta/5, \alpha, \beta)$

if $(a + \alpha/5) \leq (b - \beta/5)$

⇨ NORM(CONTR(X)) otherwise

extremely X = CONTR(CONTR(X)) =

$(a + \alpha/2.5, b - \beta/2.5, \alpha, \beta)$

if $(a + \alpha/2.5) \leq (b - \beta/2.5)$

⇨ NORM(CONTR(CONTR(X))) otherwise

more_or_less X ⇨ $DIL(X) = (a - \alpha/5, b + \beta/5, \alpha, \beta)$

indeed X ⇨ $INT(X) = (a - \alpha/4, b + \beta/4, \alpha/2, \beta/2)$

hardly X ⇨ $FUZ(X) = (a + \alpha/4, b - \beta/4, \alpha/2, \beta/2)$

if $(a + \alpha/4) < (b - \beta/4)$

⇨ $(a + \alpha(b-a)/(\alpha + \beta), (b - \beta(b-a)/(\alpha + \beta), \alpha/2, \beta/2)$

otherwise

Remaining Definitions (cont.)

younger_than X ⇒ SM(X) = (w, a - α, 0, α)

older_than X ⇒ GR(X) = (b + β, z, β , 0)

from_to (X)(Y) ⇒ (a, d, α , δ)

in_between (X)(Y) ⇒ *from_to* (GR(X))(SM(Y))
⇒ (b + β , c- γ , β , γ)

if necessary, normalize X before applying SM(X) or GR(X).

Defuzzification Methods

- **COG (Center of Gravity)**
- **MOM (Mean of Maxima)**

Defuzzification Methods (COG)

- Center-of-Gravity Method

The center of gravity (or centroid of the distribution) is obtained by computing the *first moment* of $X(z) \subseteq U$:

$$COG\{X(z)\} = \frac{\int_U zX(z)dz}{\int_U X(z)dz} = \frac{\int_U zX(z)dz}{|X|}$$

For the discrete and finite case, where U={$z_{\mathbf{1}}$, $z_{\mathbf{2}}$, ... , $z_{\mathbf{n}}$}, this expression reduces to:

$$COG\{X(z)\} = \frac{\sum_{i=1}^{n} z_i X(z_i)}{\sum_{i=1}^{n} X(z_i)} = \frac{\sum_{i=1}^{n} z_i X(z_i)}{|X|}$$

Defuzzification Methods (MOM)

- Mean Of Maxima

The MOM method obtains the *mode* of the distribution. i.e the value of z in which the membership distribution $X(z) \subseteq U$ achieves its maximum.

If the maximum is obtained in multiple points, the output is the average of such set of points.

We first define the crisp subset V (for the continuous and discrete cases) as :

$$V(z) = \{z \in U \mid X(z) \geq X(t), \forall t \in U\}$$

$$V(z_i) = \{z_i \in U \mid X(z_i) = SUP_{j=1,\ldots,n} X(z_j)\}$$

Defuzzification Methods (MOM)

Then the MOM for the continuous case is :

$$MOM\{X(z)\} = \frac{\int_V zX(z)dz}{\int_V X(z)dz} = \frac{\int_V zdz}{\int_V dz} = \frac{\int_V zdz}{|V|}$$

And for the discrete and finite case where $U=\{z_1, z_2, \ldots, z_n\}$, this expression reduces to the average of the points in V:

$$MOM\{X(z_i)\} = \frac{\sum_{z_i \in V} z_i X(z_i)}{\sum_{z_i \in V} X(z_i)} = \frac{\sum_{z_i \in V} z_i}{\sum_{z_i \in V} 1} = \frac{\sum_{z_i \in V} z_i}{|V|}$$