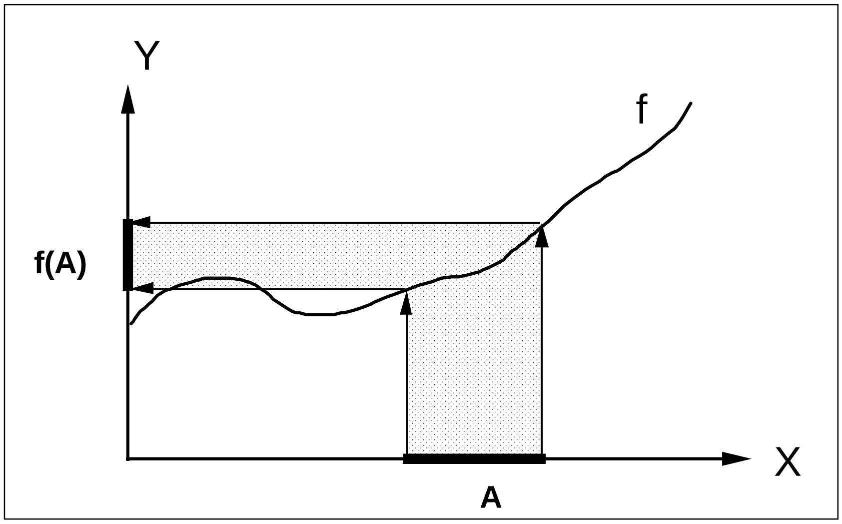
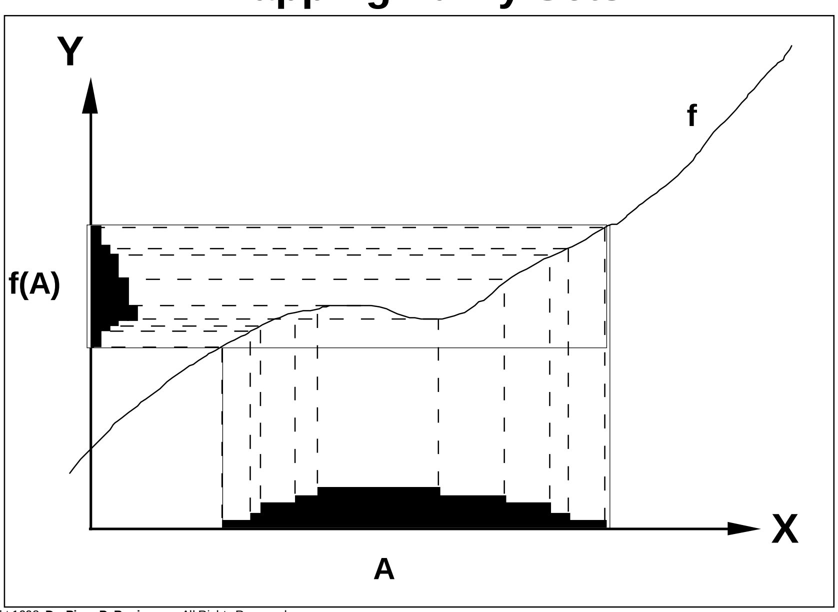
Mapping Conventional Sets



Mapping Fuzzy Sets



The Extension Principle

- Permits generalization of conventional operators
- Based on the generalization of a function

  f: X → Y

  into a function mapping fuzzy subsets of **X** into fuzzy subsets of **Y**
- If **x** has a degree of membership μ , then **y = f(x)** is assigned a degree of membership μ

  [If more than one **x** is mapped into **y** then the maximum of such memberships is used as the definition of the degree of membership of **y**]

Fuzzy Numbers

- Special fuzzy subsets of the real line
- Examples:
  - Approximately 6
  - Very large
  - Small
- May be combined using generalized operations, e.g.,

$$(A+B)(z) = \sup_{z=x+y} \{ \min [A(x), B(y)] \}$$

The Extension Principle (continued)

$$f : X_1 \times X_2 \times \cdots X_n \rightarrow Y,$$
$$f(A_1, A2, \ldots, A_n)(y) = \sup_{f(x_1, x_2, \ldots, x_n)=y} \min_i [A_i(x_i)]$$

Example:

A = 0.1/1 + 0.2/2 + 1/3 + 0.1/4

B = 0.3/1 + 1/2 + 0.5/3

A+B = 0.1/2 + 0.2/3 + 0.3/4 + 1/5 + 0.5/6 + 0.1/7

The Extension Principle (continued)

$$f : X_1 \times X_2 \times \cdots X_n \rightarrow Y,$$
$$f(A_1, A2, \ldots, A_n)(y) = \sup_{f(x_1, x_2, \ldots, x_n)=y} \min_i [A_i(x_i)]$$

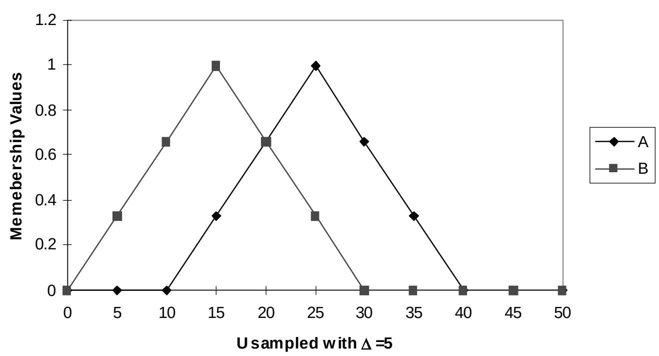
- a) Problem with sampling rate
  - Under-valuation v. increasing complexity
- b) Sampling Ordinate instead of Abscissa
- c) Parametric Representation
  - L-R Fuzzy Numbers (3 parameters)
  - L-R Fuzzy Intervals (4 parameters)
  - Mixed Arithmetics

Example of Two Fuzzy Numbers (Δ= 5)

U = [0,100]

U Sampled1 (Δ =5)	0	5	10	15	20	25	30	35	40	45	50
A	**0**	**0**	**0**	**0.33**	**0.66**	**1**	**0.66**	**0.33**	**0**	**0**	**0**
B	**0**	**0.33**	**0.66**	**1**	**0.66**	**0.33**	**0**	**0**	**0**	**0**	**0**

Low Sampling Rate for Fuzzy Numbers A (~25) and B (~15)



Example (Cont.) - a) Problem with Sampling Rate

- Let's use the Extension Principle to compute the result at one point, z=(x+y) =25, when sampling period Δ is 5

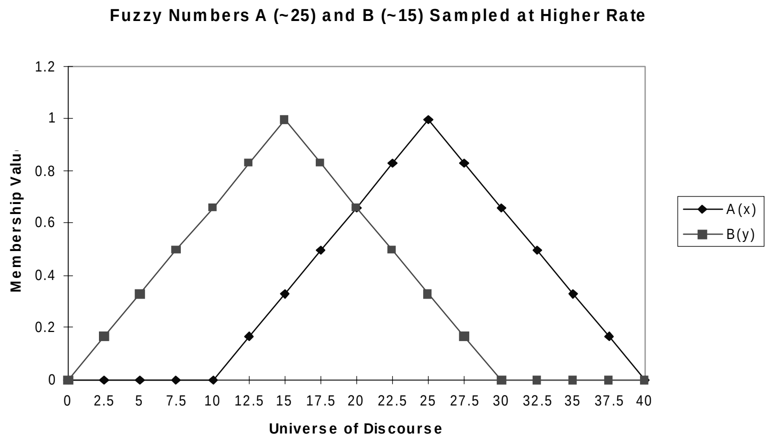
U Sampled1 (Δ =5)	0	5	10	15	20	25	30	35	40	45	50
A	**0**	**0**	**0**	**0.33**	**0.66**	**1**	**0.66**	**0.33**	**0**	**0**	**0**
B	**0**	**0.33**	**0.66**	**1**	**0.66**	**0.33**	**0**	**0**	**0**	**0**	**0**
A+B											
z= (x+y) = 25											
0+25						0					
5+20						0					
10+15						0					
15+10						0.33					
20+5						0.33					
25+0						0					
Max (Min(A(x),B(y)) | z=(x+y)						0.33					

min{ A(x), B(y) }

Example (cont.)= Increasing Sampling rate (Δ = 2.5)

U	0	2.5	5	7.5	10	12.5	15	17.5	20	22.5	25
A(x)	**0**	**0**	**0**	**0**	**0**	**0.17**	**0.33**	**0.495**	**0.66**	**0.83**	**1**
B(y)	**0**	**0.17**	**0.33**	**0.495**	**0.66**	**0.83**	**1**	**0.83**	**0.66**	**0.5**	**0.33**

o o o



Example (Cont.) - Increasing Sampling Rate

- Let's increase the sampling rate (Δ = 2.5) and compute the result at one point, z=(x+y) =25

U Sampled1 (Δ = 2.5)	0	2.5	5	7.5	10	12.5	15	17.5	20	22.5	25
A	**0**	**0**	**0**	**0**	**0**	**0.17**	**0.33**	**0.495**	**0.66**	**0.83**	**1**
B	**0**	**0.17**	**0.33**	**0.495**	**0.66**	**0.83**	**1**	**0.83**	**0.66**	**0.5**	**0.33**

A+B **z= (x+y) = 25**	
0+25	0
2.5+22.5	0
5+20	0
7.5+17.5	0
10+15	0
12.5+12.5	0.17
15+10	0.33
17.5+12.5	0.5
20+5	0.33
22.5+2.5	0.17
25+0	0
Max (Min(A(x),B(y)) | z=(x+y)	**0.5**

min{ A(x), B(y) }

Caveat

- However, increasing sampling rates implies increasing complexity of all operations, especially Modus Ponens

- Compare in the following tables the storage complexity for D = 11 and D =21:
  - D = 11 n =2 t = 7 |Rt| = 1,331
  - D = 21 n = 2 t =7 |Rt| = 9,261

Analysis of Storage Complexity for Generalized Modus Ponens - Table 1 (D=11)

For D = m =11 (Sampling Points)

Inputs	Values	Rules	Active Rules	Size of Cartesian Product	Size of Rule Table
n	**t**	**r= t^n**	**Ar= 2^n**	**$\lvert Rt \rvert$ = m D^n**	**$\lvert T \rvert$ = (nD+m)r**
2	3	9	4	1,331	297
2	5	25	4	1,331	825
2	7	49	4	1,331	1,617
2	9	81	4	1,331	2,673
3	3	27	8	14,641	1,188
3	5	125	8	14,641	5,500
3	7	343	8	14,641	15,092
3	9	729	8	14,641	32,076
4	3	81	16	161,051	4,455
4	5	625	16	161,051	34,375
4	7	2401	16	161,051	132,055
4	9	6561	16	161,051	360,855
5	3	243	32	1,771,561	16,038
5	5	3125	32	1,771,561	206,250
5	7	16807	32	1,771,561	1,109,262
5	9	59049	32	1,771,561	3,897,234
6	3	729	64	19,487,171	56,133
6	5	15625	64	19,487,171	1,203,125
6	7	117649	64	19,487,171	9,058,973
6	9	531441	64	19,487,171	40,920,957

Analysis of Storage Complexity for Generalized Modus Ponens - Table 2 (D=21)

For D = m =21 (Sampling Points)

Inputs	Values	Rules	Active Rules	Size of Cartesian Product	Size of Rule Table
n	t	r= t^n	Ar= 2^n	|Rt| = m D^n	|T| = (nD+m)r
2	3	9	4	9,261	567
2	5	25	4	9,261	1,575
2	7	49	4	9,261	3,087
2	9	81	4	9,261	5,103
3	3	27	8	194,481	2,268
3	5	125	8	194,481	10,500
3	7	343	8	194,481	28,812
3	9	729	8	194,481	61,236
4	3	81	16	4,084,101	8,505
4	5	625	16	4,084,101	65,625
4	7	2401	16	4,084,101	252,105
4	9	6561	16	4,084,101	688,905
5	3	243	32	85,766,121	30,618
5	5	3125	32	85,766,121	393,750
5	7	16807	32	85,766,121	2,117,682
5	9	59049	32	85,766,121	7,440,174
6	3	729	64	1,801,088,541	107,163
6	5	15625	64	1,801,088,541	2,296,875
6	7	117649	64	1,801,088,541	17,294,403
6	9	531441	64	1,801,088,541	78,121,827

Example (Cont.) - b) Sampling Ordinate

- Let's look at the analytic expressions for A and B

	0	x<10
	1/15*(x-10)	10<x<25
A(x) =	1 -1/15 * (x-25)	25<x<40
	0	x>40

	0	y<0
	1/15*(y)	0<y<15
B(y) =	1-1/15*(y-15)	15<y<30
	0	y>30

$A_L(x)$	$A_R(x)$	$B_L(y)$	$B_R(y)$
x=10+15A	x=15*(1-A)+25	y=15B	y=15(1-B)+15

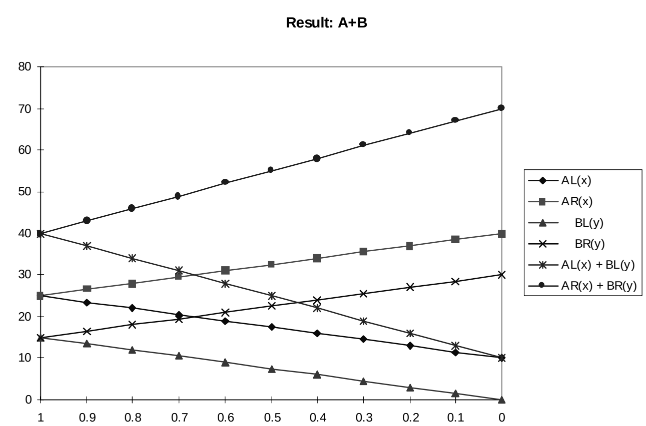
Example (Cont.) - b) Sampling Ordinate

- Let's sample the ordinate instead of the abscissa

Mem. Value	$A_L(x)$ x=10+15A	$A_R(x)$ x=15*(1-A)+25	$B_L(y)$ y=15B	$B_R(y)$ y=15(1-B)+15	$A_L(x) + B_L(y)$ x+y	$A_R(x) + B_R(y)$ x+y
1	25	25	15	15	40	40
0.9	23.5	26.5	13.5	16.5	37	43
0.8	22	28	12	18	34	46
0.7	20.5	29.5	10.5	19.5	31	49
0.6	19	31	9	21	28	52
0.5	17.5	32.5	7.5	22.5	25	55
0.4	16	34	6	24	22	58
0.3	14.5	35.5	4.5	25.5	19	61
0.2	13	37	3	27	16	64
0.1	11.5	38.5	1.5	28.5	13	67
0	10	40	0	30	10	70

Example (Cont.) - b) Sampling Ordinate

- Let's sample the ordinate instead of the abscissa



Notes

- By sampling the ordinate we do not have to perform the minimum of the membership values (they are identical)
- This is equivalent to
  - Taking α-cuts - one for each sampling value,
  - Use interval arithmetics on each of the resulting boolean intervals:
    - [a1, a2] + [b1, b2] = [a1+b1, a2+b2]
  - Reconstruct the result using the decomposing theorem

Example (cont.): c) Parametric Representation

- It' s easier to assume that there is a function for the Left and the Right slope of each fuzzy numbers.
- Each function is parametrized
- We only deal with the parameters
- We ASSUME closure of the representation under the operations performed

Membership Function (MF) Formulation

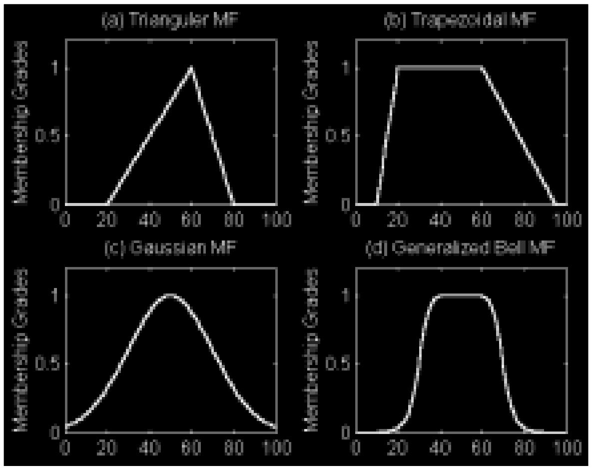
Triangular MF: $$trimf(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

Trapezoidal MF: $$trapmf(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

Gaussian MF: $$gaussmf(x;c,\sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$

Generalized bell MF: $$gbellmf(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{b}\right|^{2b}}$$

MF Formulation



disp_mf.m

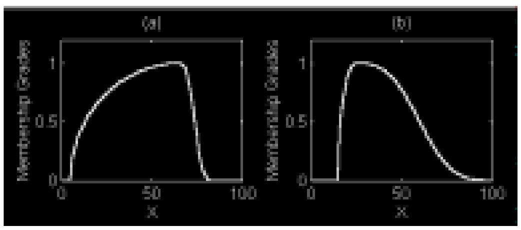
MF Formulation

L-R MF:

$$LR(x; c, \alpha, \beta) = \begin{cases} F_L\left(\frac{c-x}{\alpha}\right), x < c \\ F_R\left(\frac{x-c}{\beta}\right), x \geq c \end{cases}$$

Example: $F_L(x) = \sqrt{\max(0, 1 - x^2)}$ $F_R(x) = \exp(-|x|^3)$

c=65
a=60
b=10



c=25
a=10
b=40

difflr.m

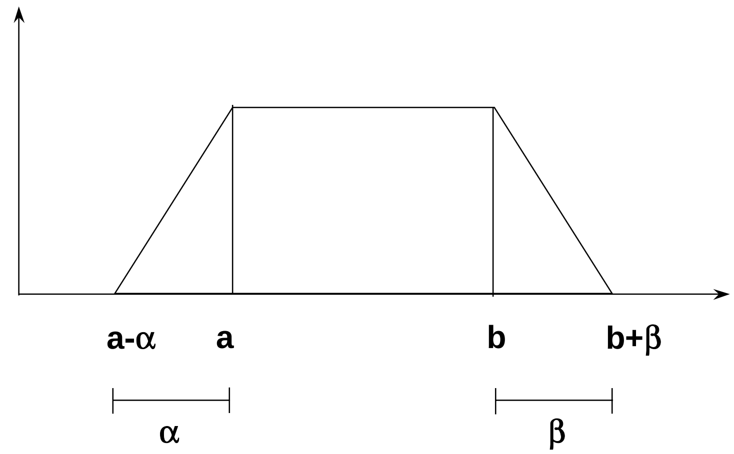
Example (cont.): c) Parametric Representation

- We will use a 4-parameter representation:

  (a, b, α, β)

- where
  - Core: [a,b]
  - Support: [(a- α), (b+ β)]
  - Slopes: Linear functions
- Valid only for CONVEX, NORMAL fuzzy intervals
- We will ASSUME closure of the representation under the operations performed (although this is true only for addition and subtraction)

Fuzzy Interval (a, b, α, β)



Example (cont.): c) Parametric Representation

- Mixed arithmetics representation with same parameters

Crisp Number x = (x, x, 0, 0)
Crisp Interval [x,y] = (x, y, 0, 0)
Fuzzy number $\tilde{x}$ = (x, x, α, β)
Fuzzy Interval $[\tilde{x}, \tilde{y}]$ = (x, y, α, β)

Example (cont.): c) Parametric Representation

- See tables and handout

Where are we now? (A simplfied Roadmap)