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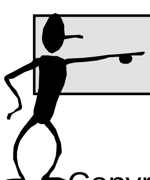


Alternative Definitions for Fuzzy-Set Intersection

- Zadeh Intersection:

$$(A \cap B)(x) = \min [A(x), B(x)]$$

- Product Intersection:

$$(A \cap B)(x) = A(x)\ \ B(x)$$

- Lukasiewicz Intersection

$$(A \cap B)(x) = \max [A(x) + B(x)-1, 0]$$

Operators that satisfy reasonable axioms for a truth-functional definition of intersection are called *triangular norms* (or *T- norms)*

Triangular Norms

$$(A \cap B)(x) = A(x) \otimes B(x)$$

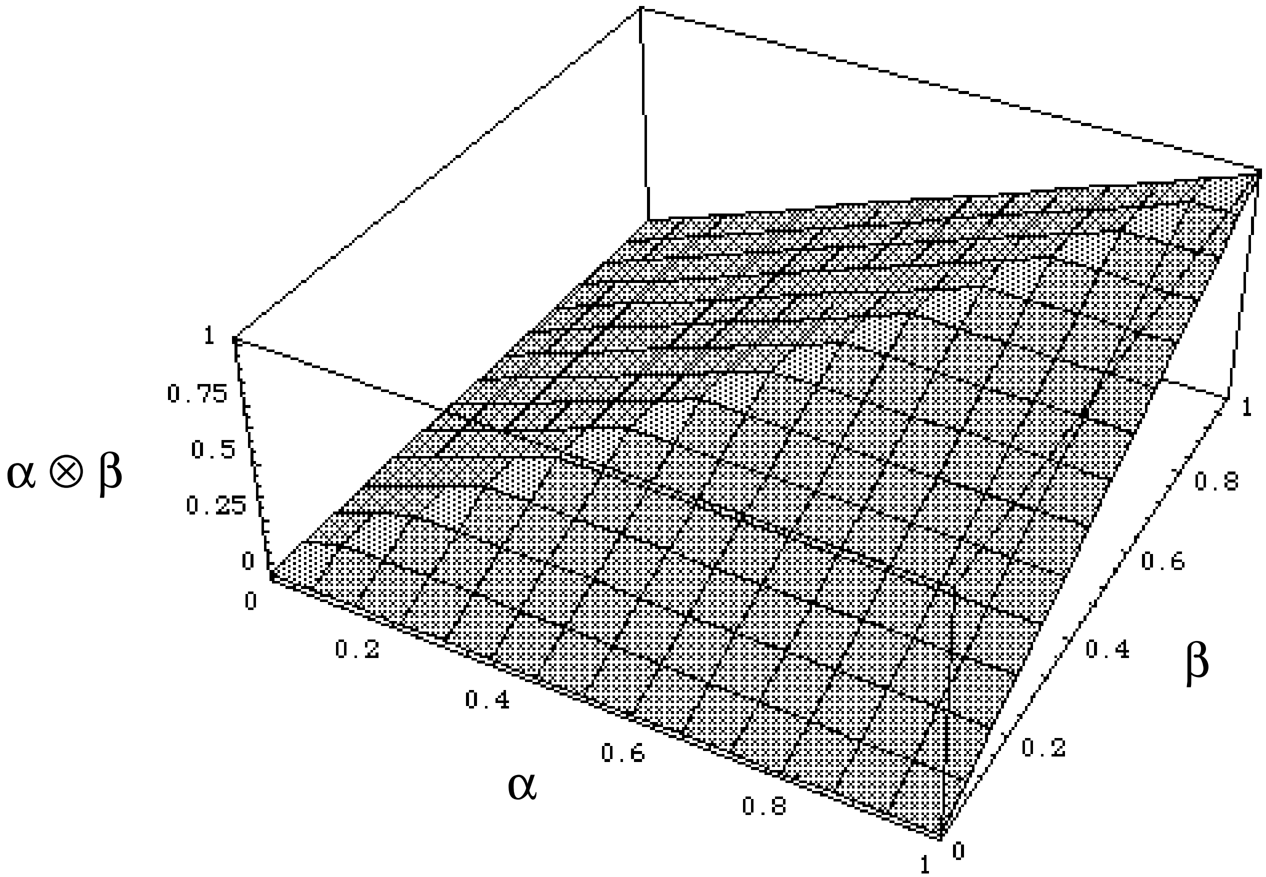
T-Norm Axioms:

- $\otimes : [0,1] \times [0,1] \rightarrow [0,1]$,
- Commutativity: $\alpha \otimes \beta = \beta \otimes \alpha$,
- Associativity: $(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$
- Monotonicity:

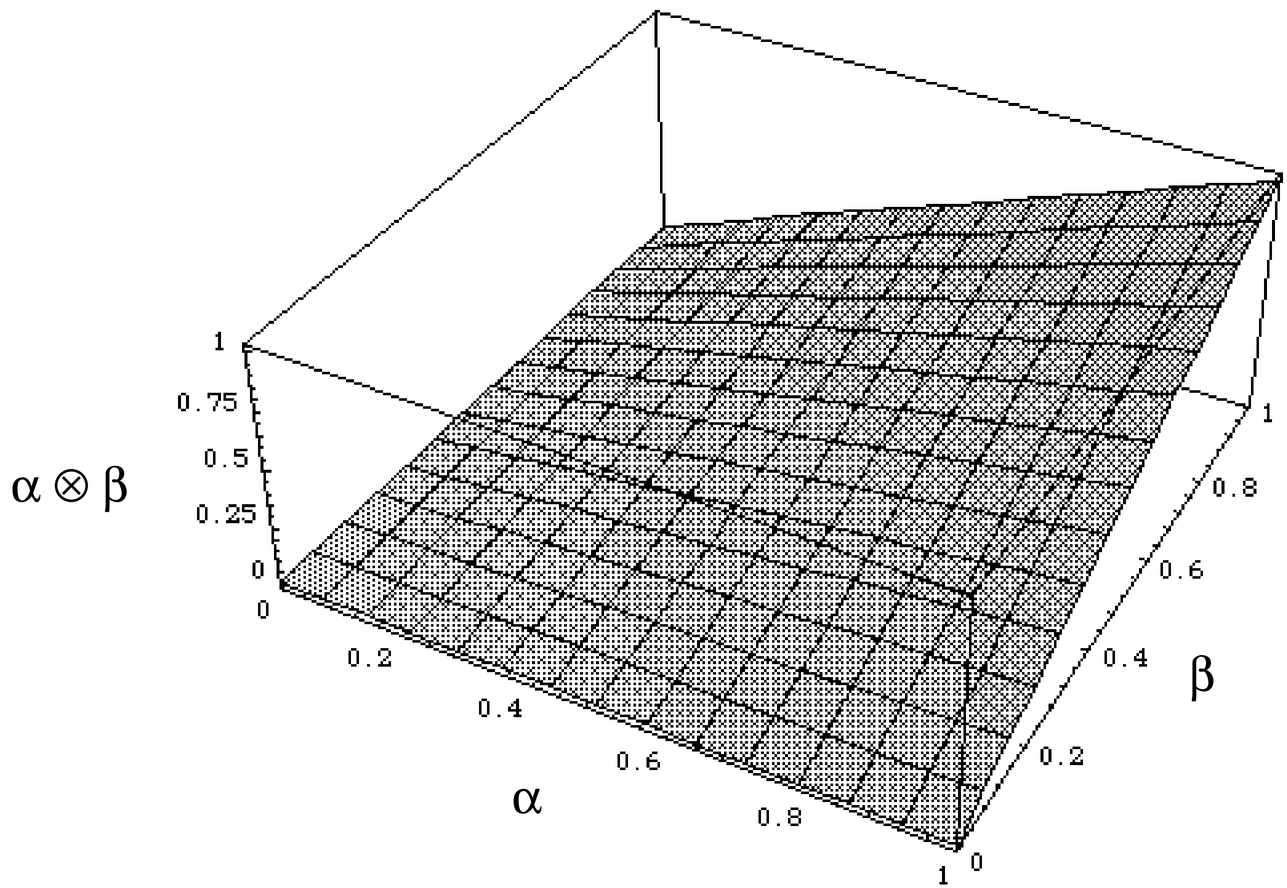
  If $\alpha \geq \alpha'$, $\beta \geq \beta'$, then $\alpha \otimes \beta \geq \alpha' \otimes \beta$, $\alpha \otimes \beta \geq \alpha \otimes \beta'$,
- Boundary Conditions

  $\alpha \otimes 1 = \alpha$, $\alpha \otimes 0 = 0$

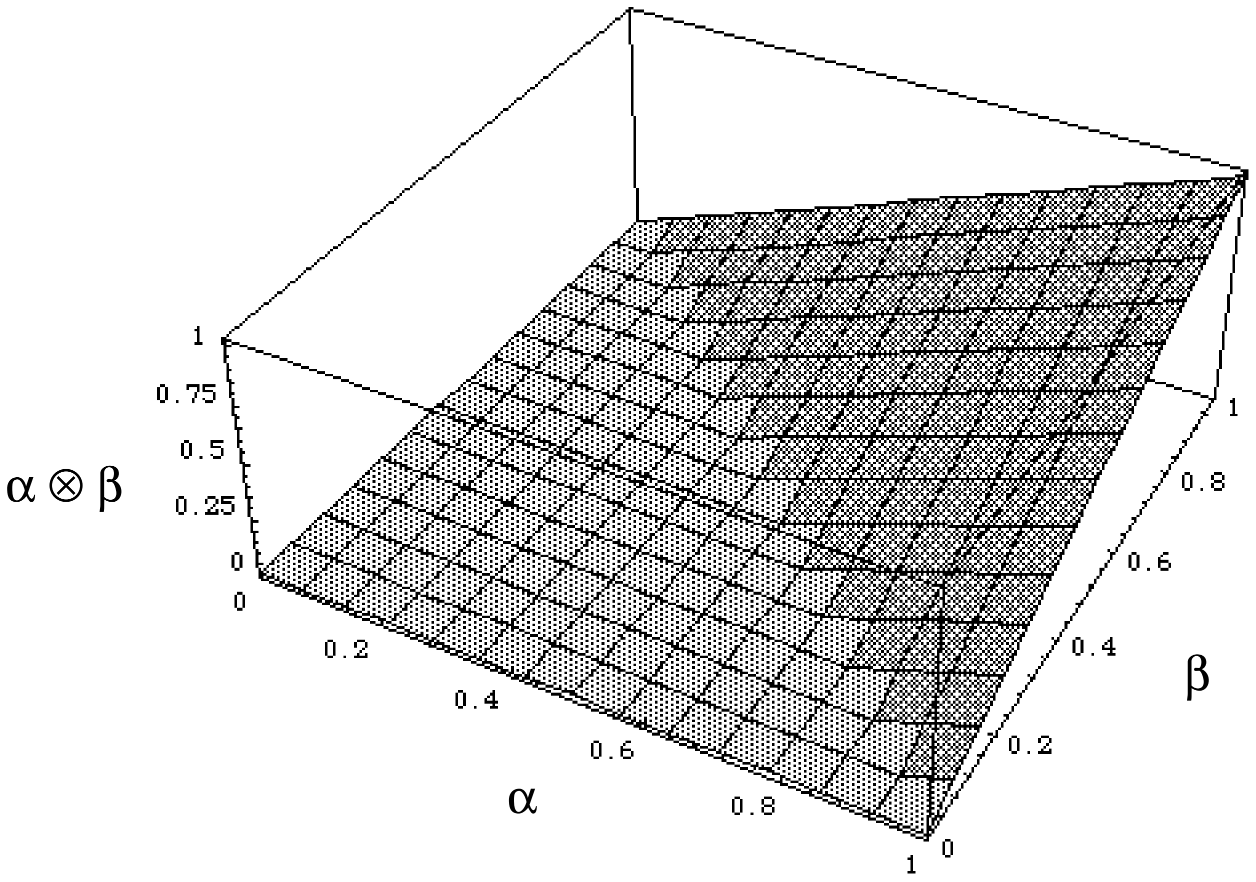
The Minimum T-Norm



The Product T-Norm



The Lukasiewicz T-Norm



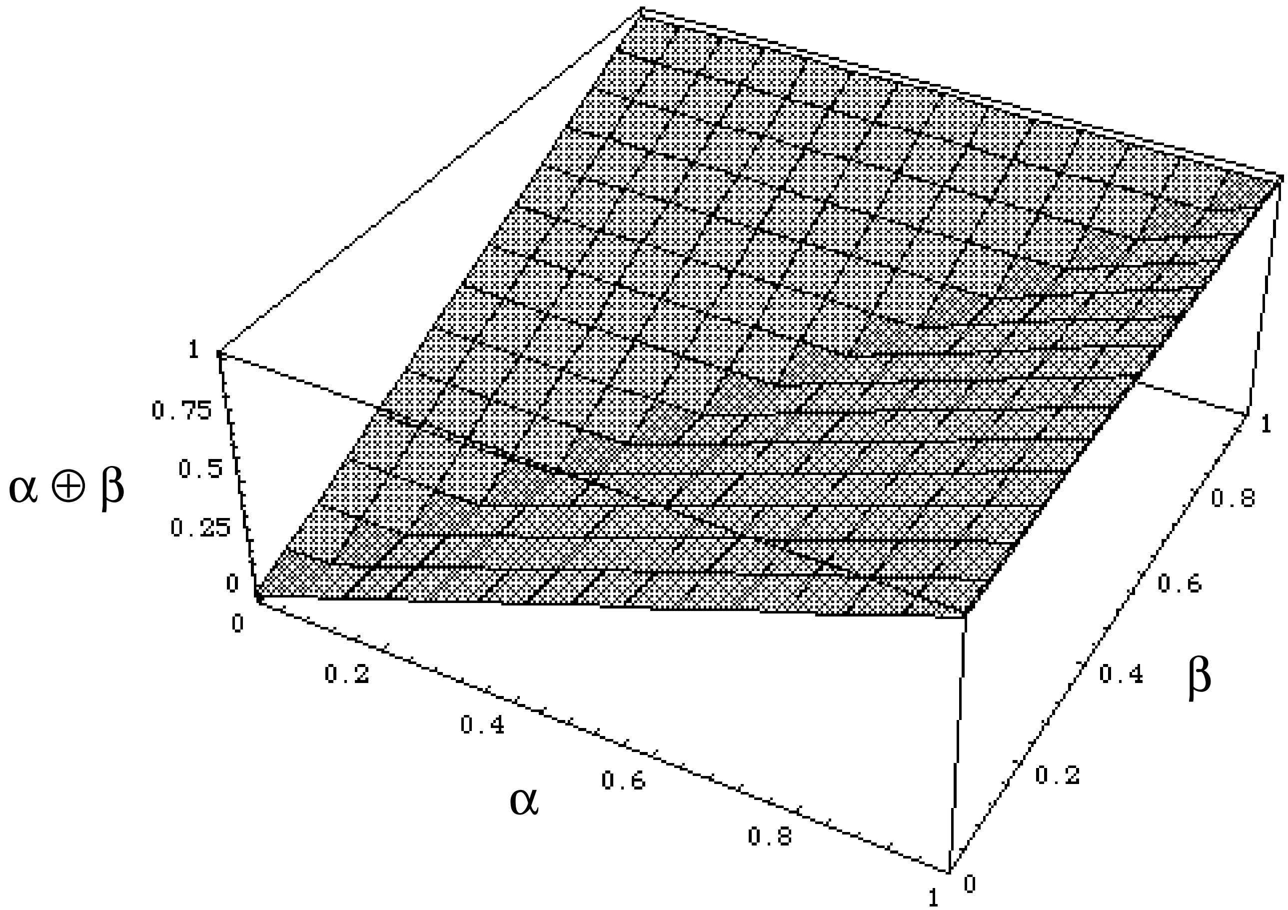
Triangular Conorms

$$(A \cup B)(x) = A(x) \oplus B(x)$$

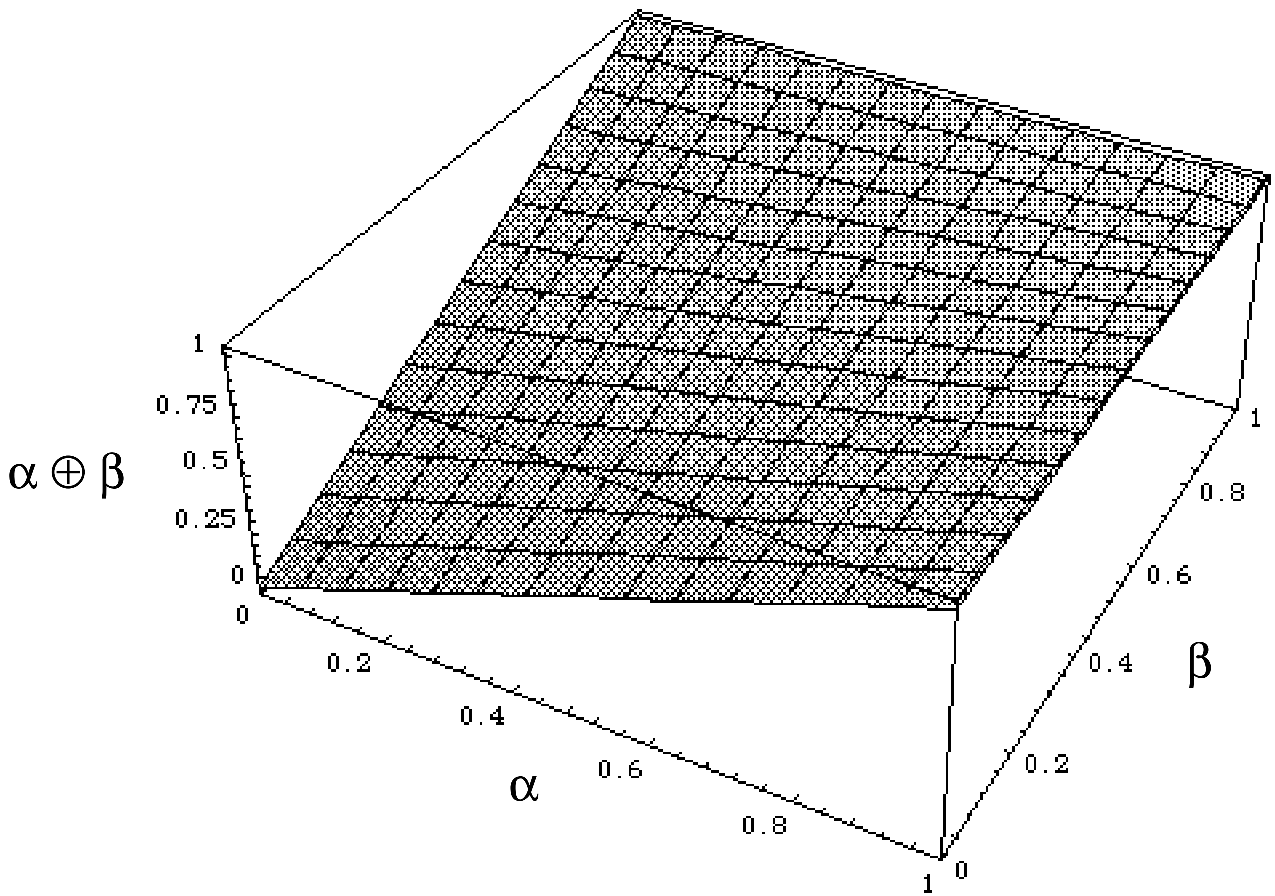
T-Conorm Axioms:

- $\oplus : [0,1] \times [0,1] \rightarrow [0,1]$,
- Commutativity: $\alpha \oplus \beta = \beta \oplus \alpha$,
- Associativity: $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$
- Monotonicity:
  If $\alpha \geq \alpha'$, $\beta \geq \beta'$, then $\alpha \oplus \beta \geq \alpha' \oplus \beta$, $\alpha \oplus \beta \geq \alpha \oplus \beta'$,
- Boundary Conditions:
  $\alpha \oplus 1 = 1$, $\alpha \oplus 0 = \alpha$.

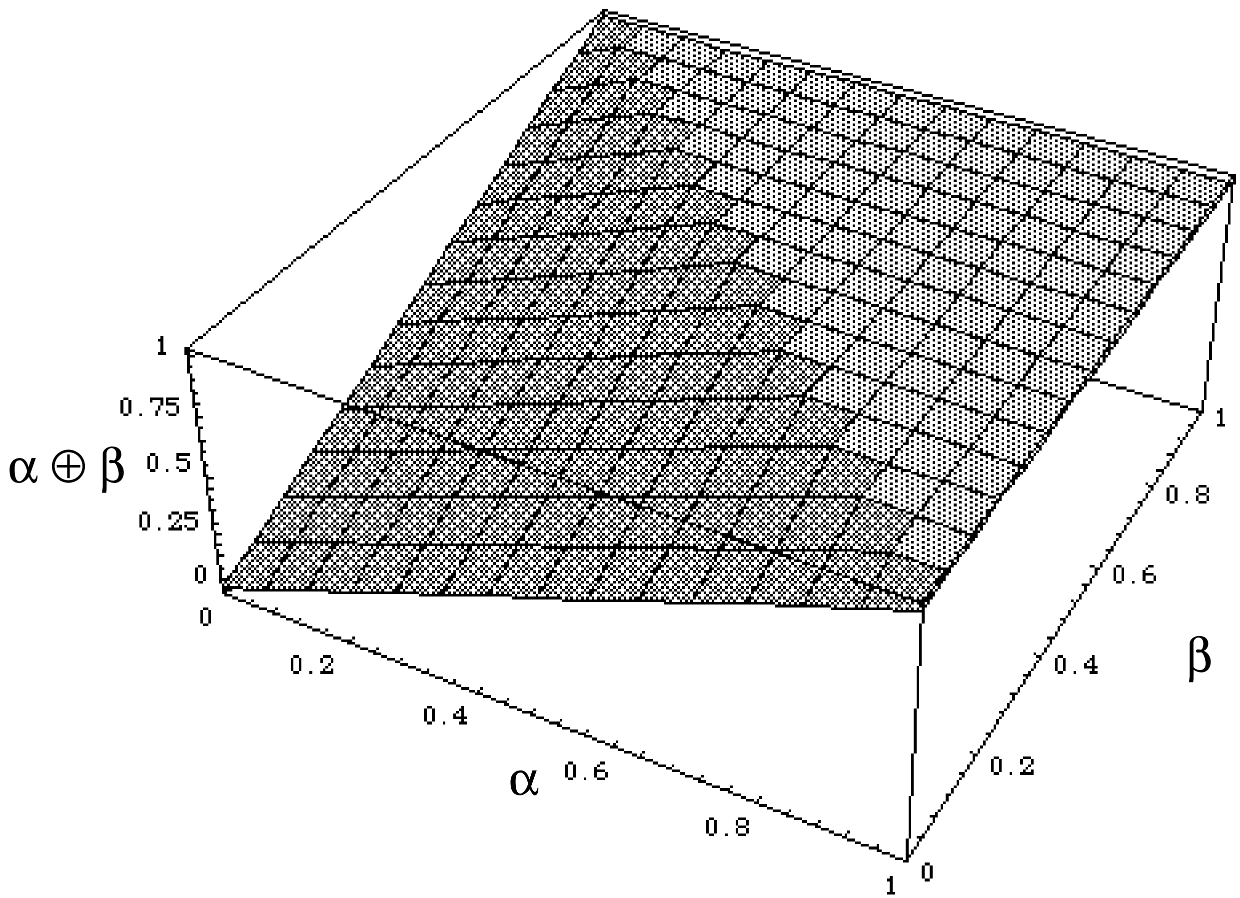
The Maximum T-Conorm



The Product T-Conorm



The Lukasiewicz T-Conorm



Fuzzy Complement

General requirements:

- **Boundary: N(0)=1 and N(1) = 0**
- **Monotonicity (non-increasing): N(a) > N(b) if a < b**
- **Involution: N(N(a) = a**

Standard fuzzy complement: $N(a) = 1 - a$

Two alternative types of fuzzy complements:

- **Sugeno's complement:**

$$N_s(a) = \frac{1-a}{1+sa}$$

- **Yager's complement:**

$$N_w(a) = (1-a^w)^{1/w}$$

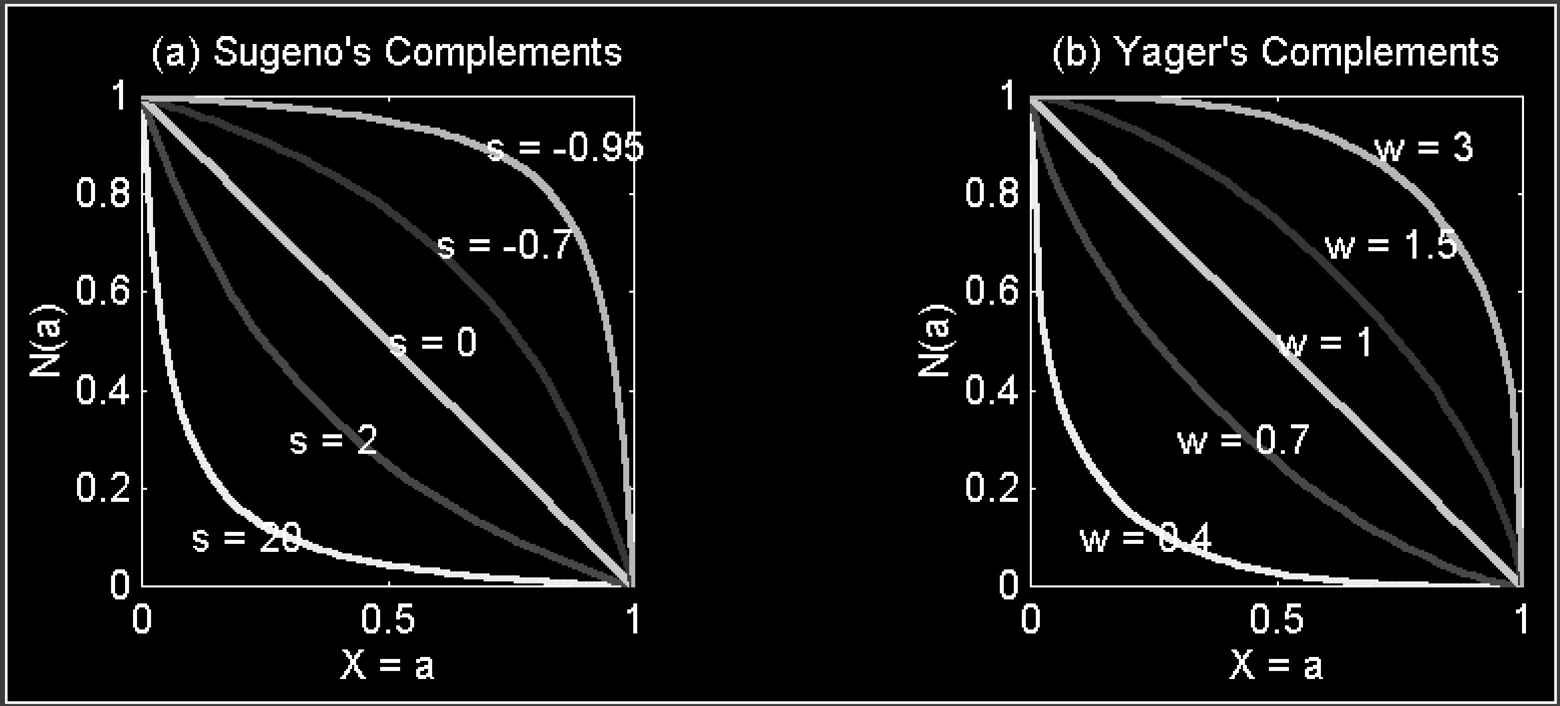
Fuzzy Complement

Sugeno's complement:

$$N_s(a) = \frac{1-a}{1+sa}$$

Yager's complement:

$$N_w(a) = (1-a^w)^{1/w}$$



negation.m

Alternative Definitions for Fuzzy-Set Union

- Zadeh Union:

$$(A \cup B)(x) = \max\,[A(x), B(x)]$$

- Product Union:

$$(A \cup B)(x) = A(x) + B(x) - A(x)\,B(x)$$

- Lukasiewicz Union

$$(A \cup B)(x) = \min\,[A(x) + B(x), 1]$$

Operators that satisfy reasonable axioms for a truth-functional definition of union are called *triangular conorms* (or *T- conorms)*

De Morgan Triples

De Morgan Laws for Conventional Sets:

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

- Triples consisting of an Intersection operator, Union operator, and Complement functions that generalize the DeMorgan Laws are called *DeMorgan triples.*

Examples:

- Zadeh: **{min(α,β), max(α,β), 1-α}**
- Product: **{$\alpha\beta$, $\alpha + \beta - \alpha\,\beta$, 1-$\alpha$}**
- Lukasiewicz: **{max(α + β -1, 0), min(α + β, 1), 1- α}**

Combinations of Operators

- Every combination of intersection, union, and complement operators fails to satisfy some property of the *Boolean Algebra* of conventional sets
- The standard operators fail to satisfy the laws *of the excluded middle* and *of contradiction:*

$$A \cap \overline{A} = \emptyset , \quad A \cup \overline{A} = X$$

- Other combinations fail to satisfy some other properties, usually those of *idempotence:*

$$A \cap A = A , \quad A \cup A = A$$

Some Basic Limitations

- If $\otimes$ is continuous and idempotent, then $\otimes =$ **min** ,
- If $\oplus$ is continuous and idempotent, then $\oplus =$ **max** ,
- Triples **{** $\otimes, \oplus, -$ **}** of continuous operators that satisfy the law of the excluded middle and contradiction are neither idempotent nor distributive
- All T-norms are bound (by above) by the **min**, i.e.,

$$\alpha \otimes \beta \leq \mathbf{min}(\alpha, \beta)$$

- All T-conorms are bound (by below) by the **max**, i.e.,

$$\alpha \oplus \beta \geq \mathbf{max}(\alpha, \beta)$$