Fuzzy Sets & Expert Systems in Computer Eng. (1):

Fuzzy Sets

Piero P. Bonissone

GE Corporate Research & Development

Bonissone@crd.ge.com

(adapted from slides by Roger Jang
and Enrique Ruspini)

Outline

- Motivation
- Fuzzy Sets Basic Concepts
  - Characteristic Function (Membership Function)
  - Examples
  - Notation
  - Semantics and Interpretations
  - Related crips sets
    - » Support, Bandwidth, Core, α-level cut
    - » Decomposition Theorem
  - Features, Properties, and More Definitions
    - » Convexity, Normality
    - » Cardinality, Measure of Fuzziness, First Moment
    - » MF parametric formulation
- Fuzzy Set-theoretic Operations
  - Intersection, Union, Complementation
  - Numerical Examples
  - T-norms and T-conorms

Energy Strategy Draws Congressional Criticism

... the Administration's national energy policy was criticized today as one sided even by those who *vigorously support* its plan for drilling in *environmentally fragile* areas.

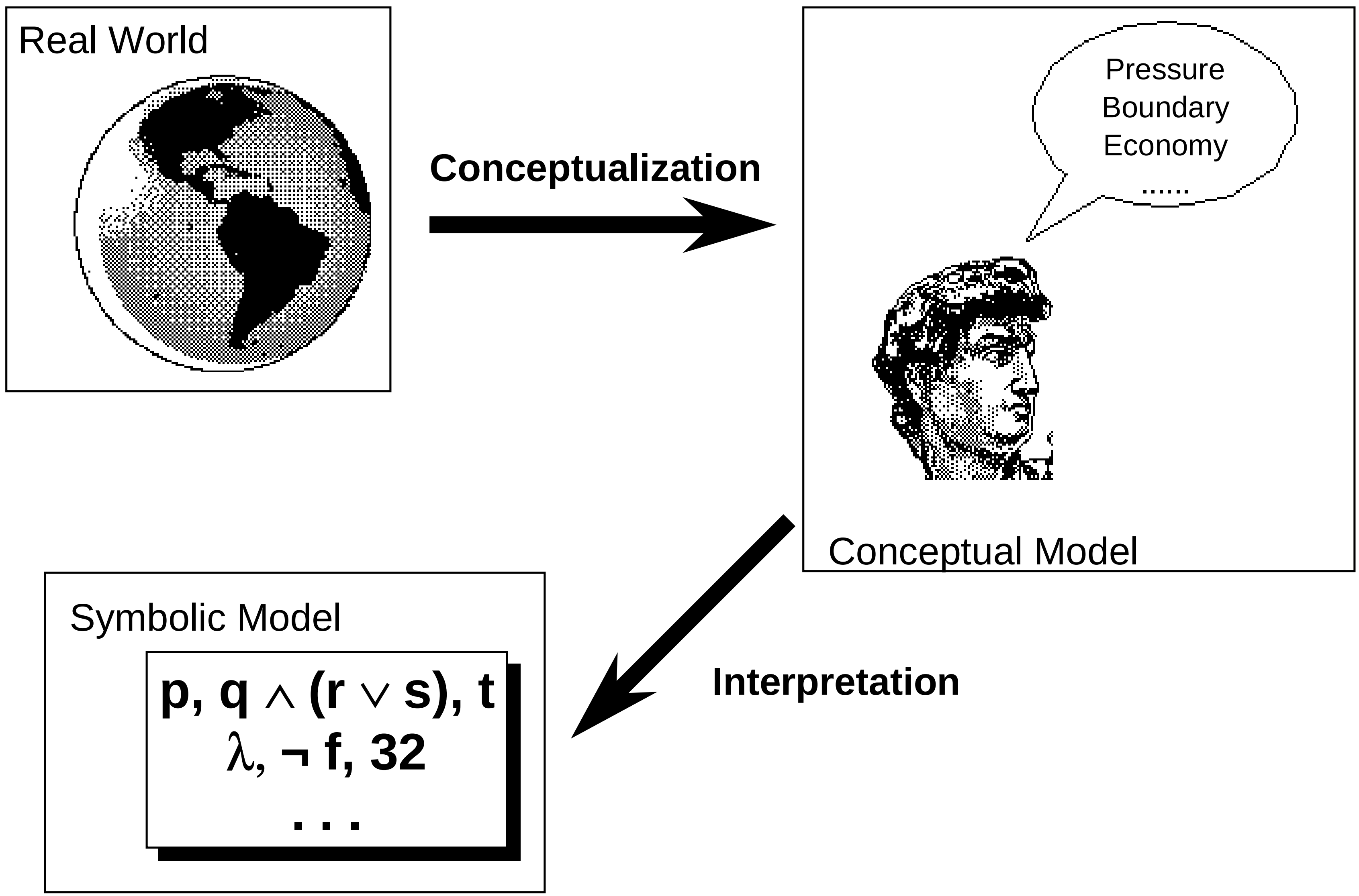
... Mr. Johnston, a *strong supporter* of *more* oil and gas drilling, as well as others forms of energy, said "You've just got to do that for *balance*; I don't care if you *believe* it or not."

New York City Population Gain Attributed to Immigrant Tide

The white population of Manhattan, including some Hispanic residents, grew by 20,263 during the decade, to 867,227, a gain of 3.1 percent. Staten Island added 8509 whites for a total of 322,043, a gain of 2.7 percent.

(New York Times, February 22, 1991)

MODELING


Real World
Conceptualization
Pressure
Boundary
Economy
......
Conceptual Model
Interpretation
Symbolic Model
p, q ∧ (r ∨ s), t
λ, ¬ f, 32
. . .

Approximate Models

Imprecise, Vague, Uncertain Representations of System Behavior

- *Classical Models:*

  If <u>Pressure</u> = 10 ATM, then <u>Volume</u> = 2.5 Cm

- *Imprecise Models:*

  If <u>Pressure</u> ≥ 5 ATM, then <u>Volume</u> ≤ 6 Cm

- *Uncertain Models:*

  If <u>Pressure</u> ≥ 5 ATM, then <u>Prob(Vol = 6 Cm)</u> = 0.9

- *Vague Models:*

  If <u>Pressure</u> is HIGH, then <u>Volume</u> is LOW

Approximate Models and Decision/Control Rules

- Vague rules may be used to describe characteristics of the system:

If Position(t) is NEAR and Velocity is HIGH, then Position(t+1) is MEDIUM,

If Shape is ROUND and Gap is SMALL, then Probability(Symbol=a) is HIGH,

- Usually, the problem-solving goal is the generation of decision and control rules:

If Position(t) is LOW and Velocity(t) is HIGH, then the Acceleration should be SMALL,

If Probability(Symbol1=a) is LOW and Probability(Symbol2=x) is HIGH, then Probability(Sequence=ex) is HIGH.

What is Fuzzy Logic ?

- Inferential Approach
- Oriented towards System Analysis/Decision Support
- Utilized to develop Intelligent Automated Systems
- Capable of dealing with Vague Information
- Facilitates development of qualitative models
- Extension of Classical Logic using Multiple Truth-Values
- Exploits notions of similarity between situations
- Based on the Theory of Fuzzy Sets (Zadeh 1965)

Why Fuzzy Logic ?

- Ability to translate imprecise/vague knowledge of human experts
- Simple, easy to implement technology
- Rules contribute to inferences even when facts do not exactly match antecedent
- Rule-based systems may be analyzed and improved
- Technology is easy to transfer from product to product
- Smooth controller behavior
- Robust controller behavior
- Ability to control unstable systems

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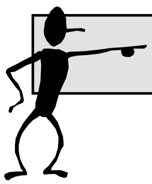

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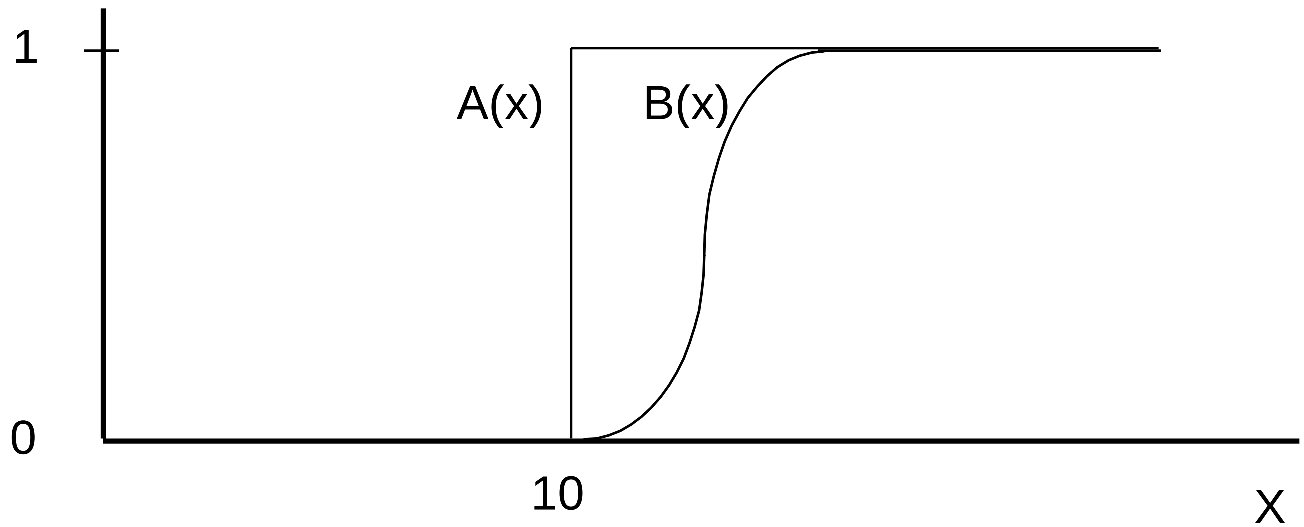
Fuzzy Sets

A = { x ∈ **X** | x **>** 10}	Boolean Set	A: **X** → {0, 1}
B = { x ∈ **X** | x **>>** 10}	Fuzzy Set	B: **X** → [0, 1]



Characteristic function of sets A(x) and B(x)

Boolean Algebra (Run through)

Assign binary truth value to statements

A	statement
1	true
0	false

A	¬ A
1	0
0	1

Combine statements using AND and OR operators

A	B	AvB
0	0	0
0	1	1
1	0	1
1	1	1

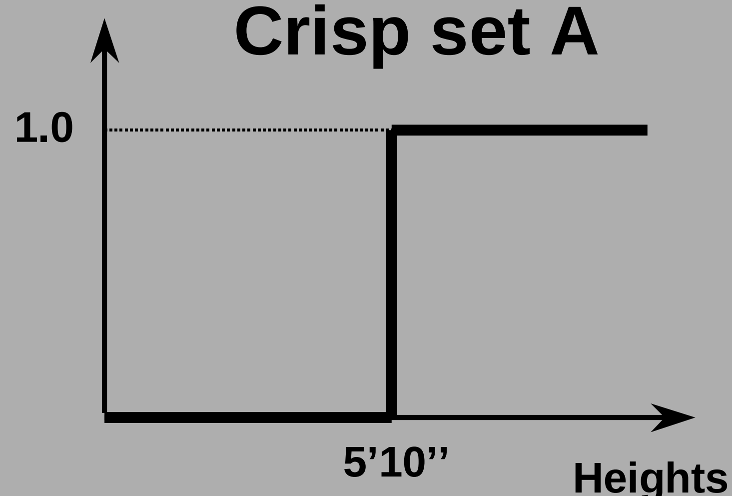
A	B	A ^ B
0	0	0
0	1	1
1	0	0
1	1	1

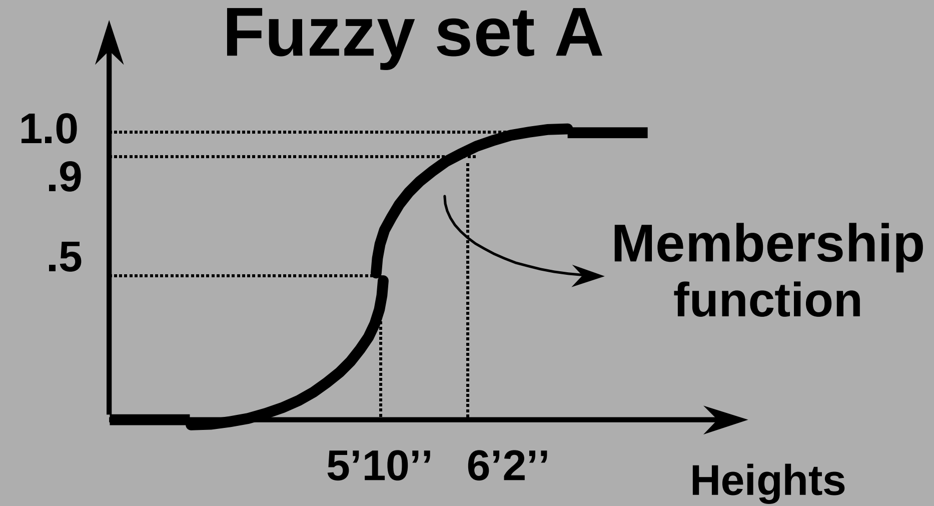
Fuzzy Sets

Binary Logic vs. Fuzzy Logic:

Sets with crisp and fuzzy boundaries, respectively

A = Set of tall people

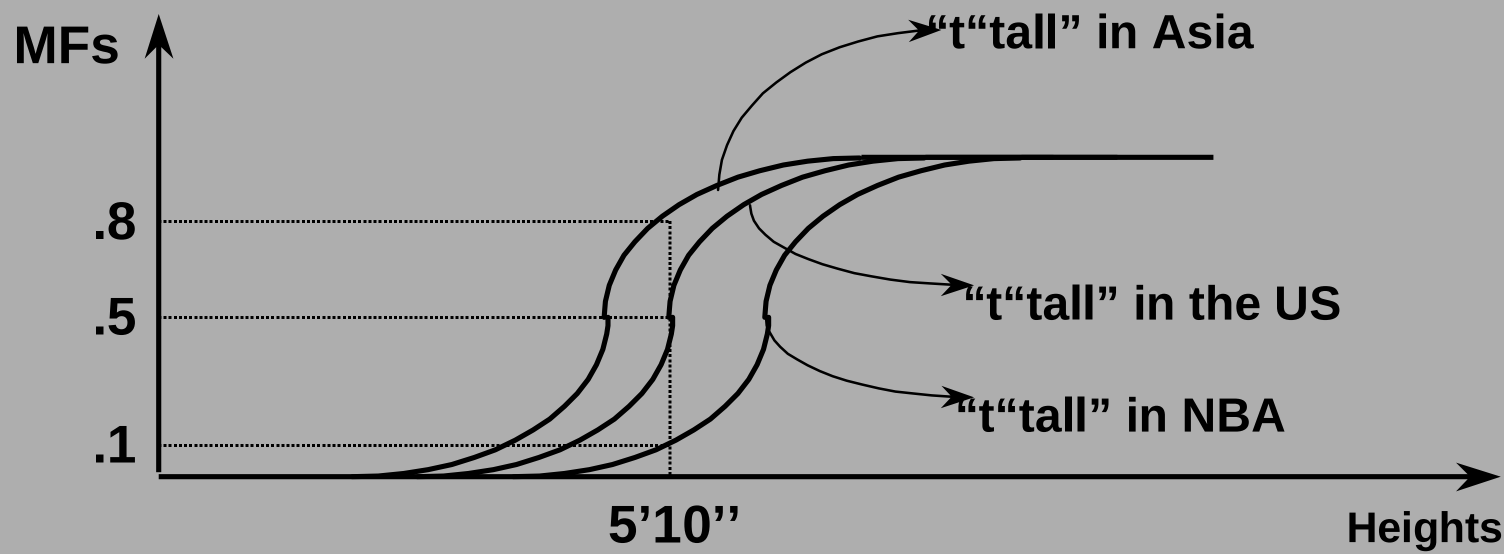




Membership Functions (MFs)

Characteristics of MFs:

- Subjective measures
- Not probability functions



Fuzzy Sets

Formal definition:

A fuzzy set *A* in *X* is expressed as a set of ordered pairs:

$$A = \{(x, \mu_A(x)) \mid x \in X, \mu_A(x): X \mapsto [0,1]\}$$

Fuzzy set — A

Membership function (MF) — $\mu_A(x)$

Universe or universe of discourse — X

A fuzzy set is totally characterized by a membership function (MF).

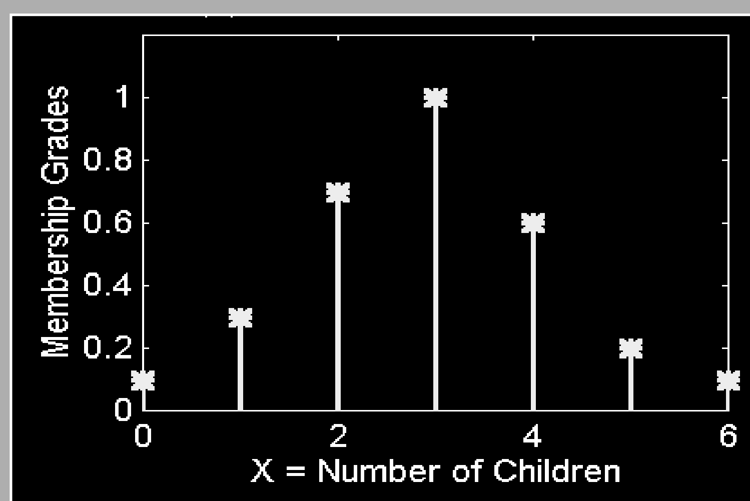
Fuzzy Sets with Discrete Universes

Fuzzy set C = "desirable city to live in"

- **X = {SF, Boston, Troy} (discrete and nonordered)**
- **C = {(SF, 0.9), (Boston, 0.8), (Troy, 0.6)}**

Fuzzy set A = "sensible number of children"

- **X = {0, 1, 2, 3, 4, 5, 6} (discrete universe)**
- **A = {(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)}**

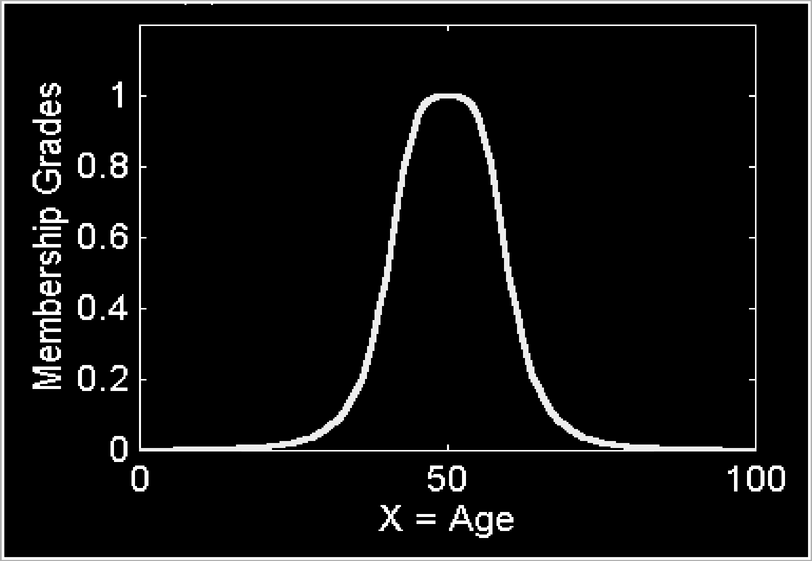


Fuzzy Sets with Cont. Universes

Fuzzy set B = "about 50 years old"

- X = Set of positive real numbers (continuous)
- B = {(x, μ_B(x)) | x in X}

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x - 50}{10}\right)^2}$$



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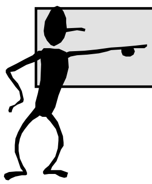

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Examples of Fuzzy Sets

- Tall Persons *(Height)*
- Dangerous Maneuvers *(Action Sequences)*
- Blonde Individuals *(Hair color)*
- Loud Noises *(Sound Intensity)*
- Large Investments *(Money)*
- High Speeds *(Speed)*
- Close Objects *(Distance)*
- Large Numbers *(Numbers)*
- Desirable Actions *(Decision or Control Space)*

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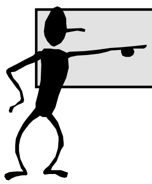
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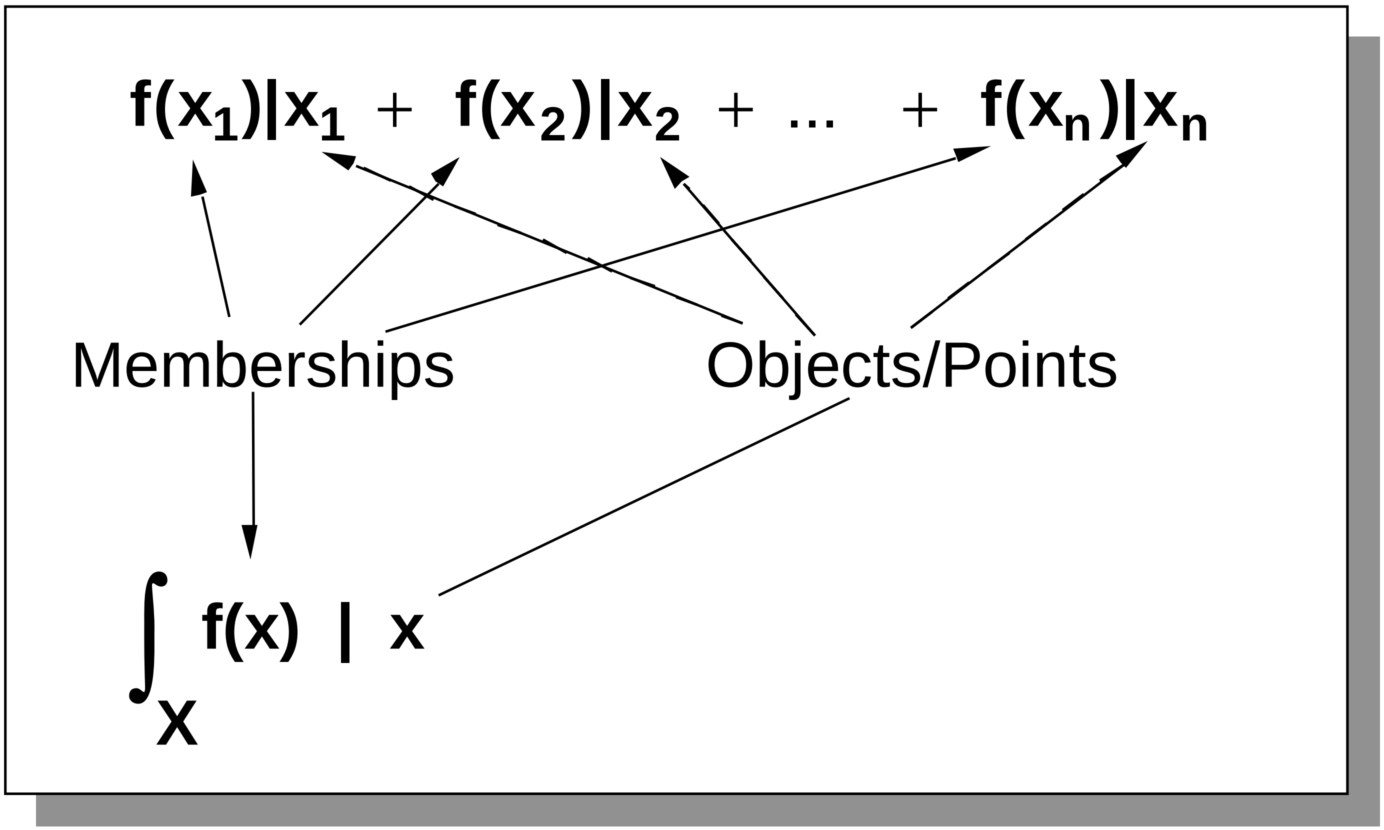
Denoting Fuzzy Sets

$$f : X \rightarrow [0,1] : x \rightarrow f(x)$$



Alternative Notation

A fuzzy set A can be alternatively denoted as follows:

X is discrete ⇨ $$A = \sum_{x_i \in X} \mu_A(x_i) / x_i$$

X is continuous ⇨ $$A = \int_X \mu_A(x) / x$$

Note that Σ and integral signs stand for the union of membership grades; "/" stands for a marker and does not imply division.

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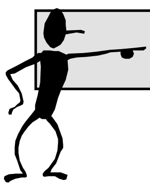

Probability vs. Fuzziness

Randomness:

Uncertainty described by tendency (frequency) of a random variable to take on a value in a specified region

Interpretations: frequency -> willingness to accept bet (subjective probability)

Fuzziness:

Degree to which the element satisfies properties characterized by a fuzzy set.

Interpretations: Possibility -> similarity -> desirability

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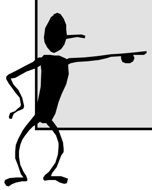

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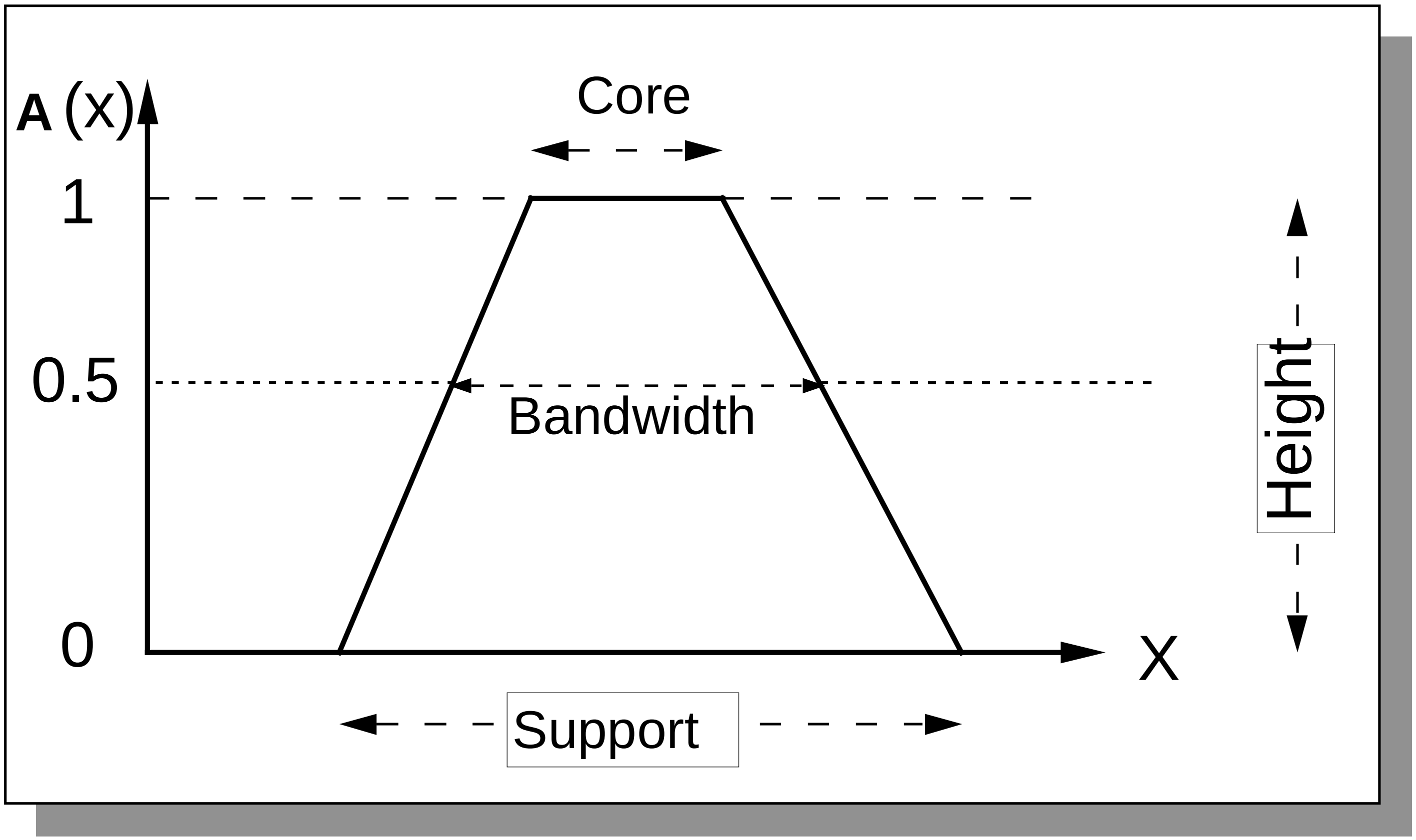
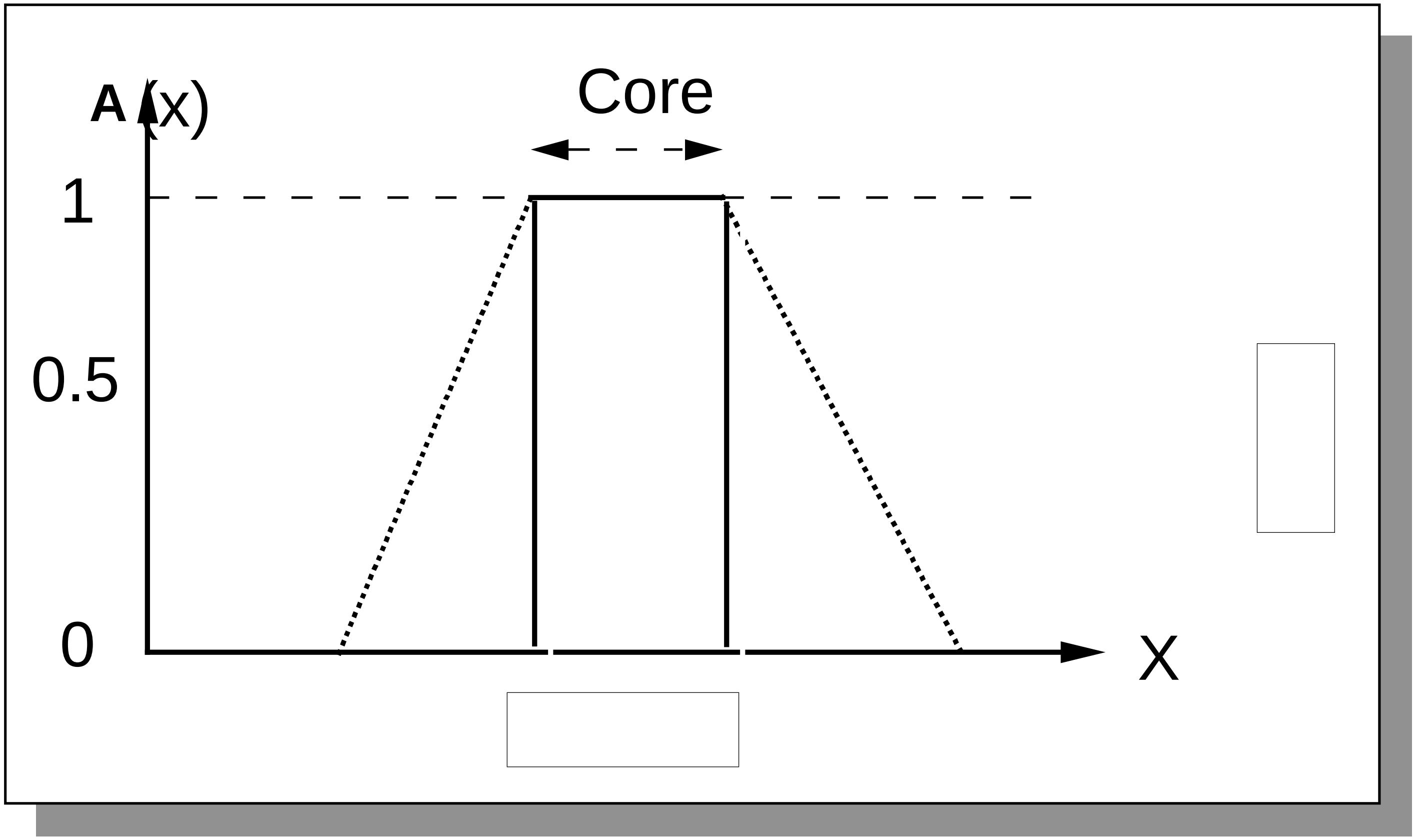
Fuzzy Sets



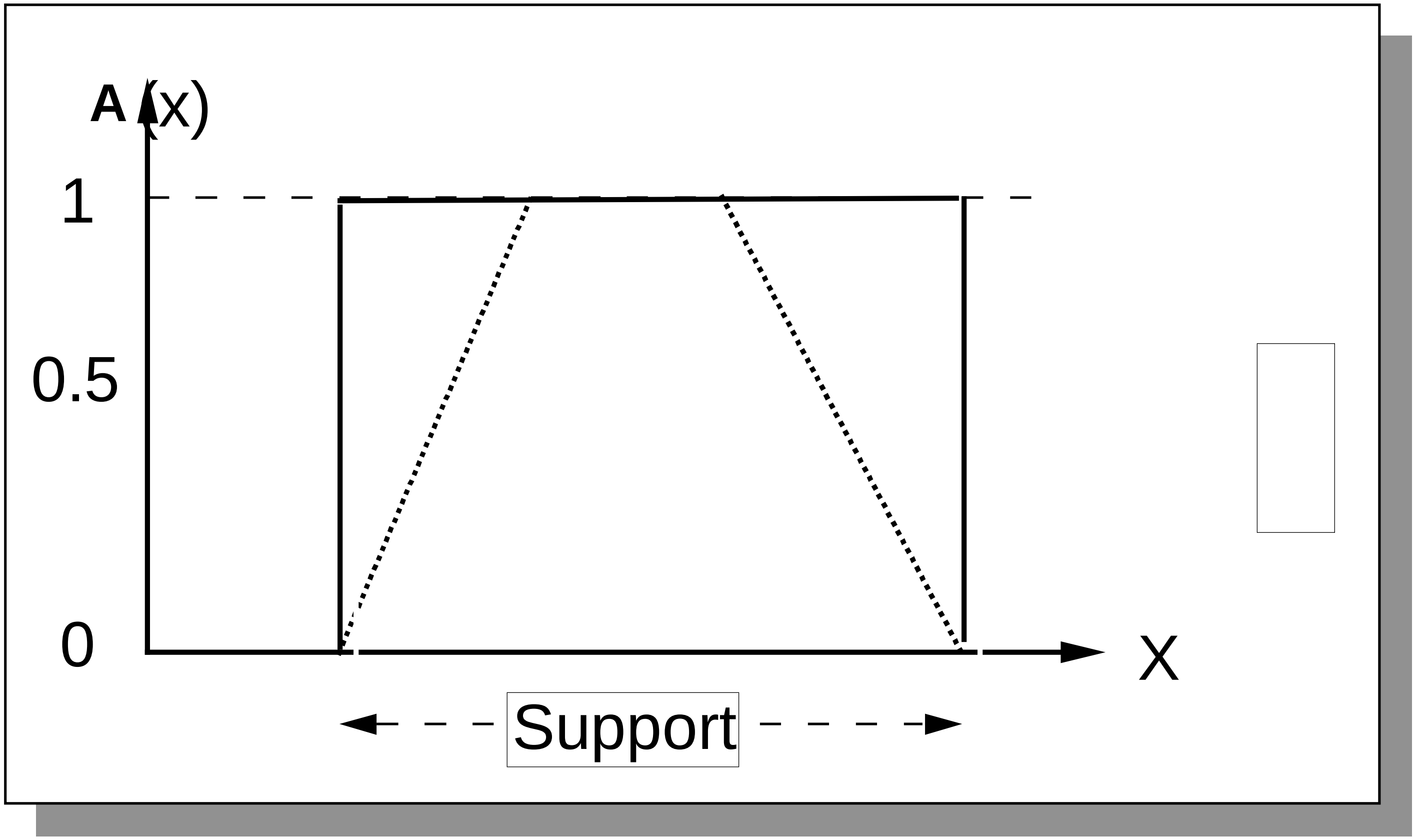
$$\textbf{Core(A)} = \{x \mid A(x) = 1\}$$
$$\textbf{Support(A)} = \{x \mid A(x) > 0\}$$

Fuzzy Sets



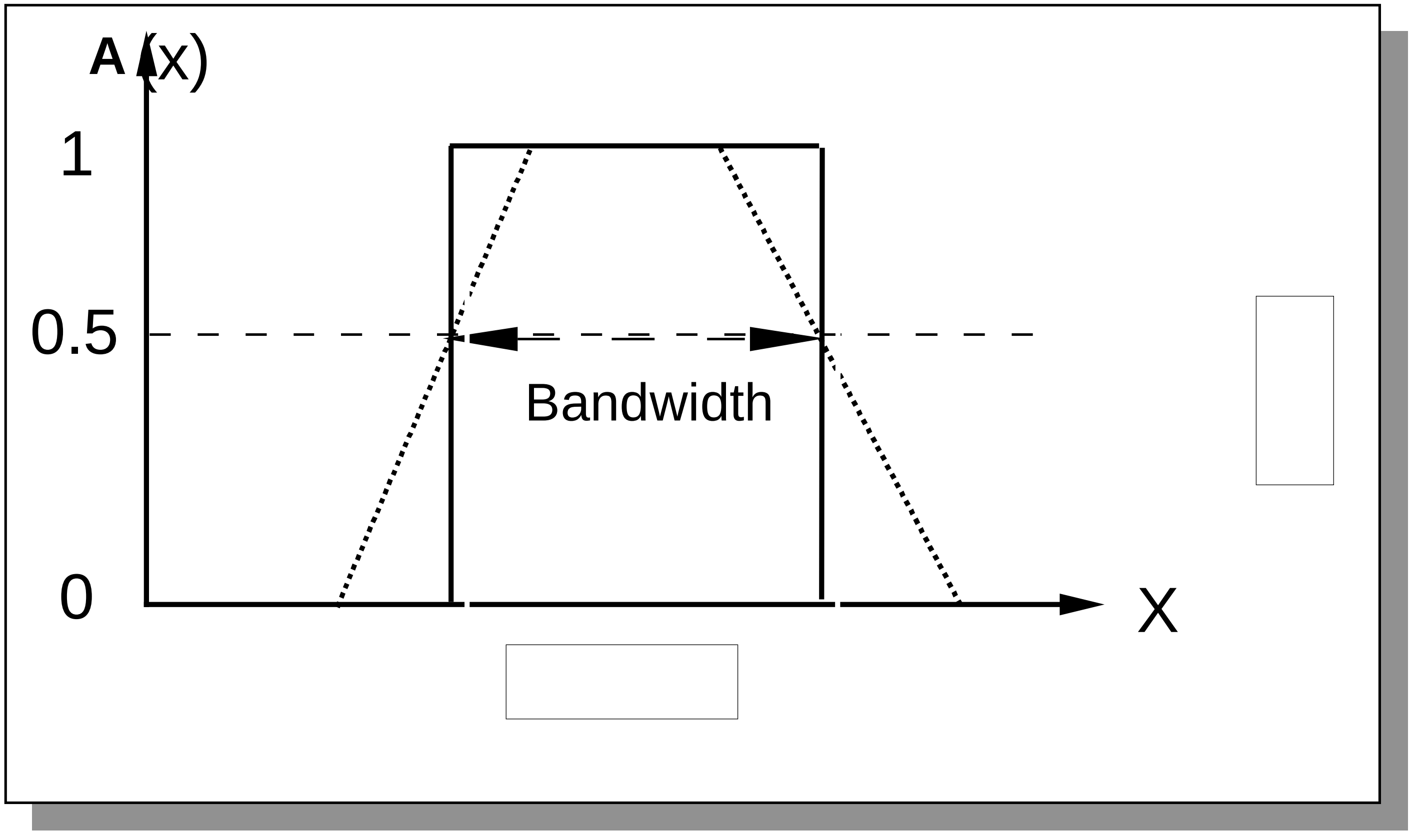
Core(A) $= \{x \mid A(x) = 1\}$

Fuzzy Sets



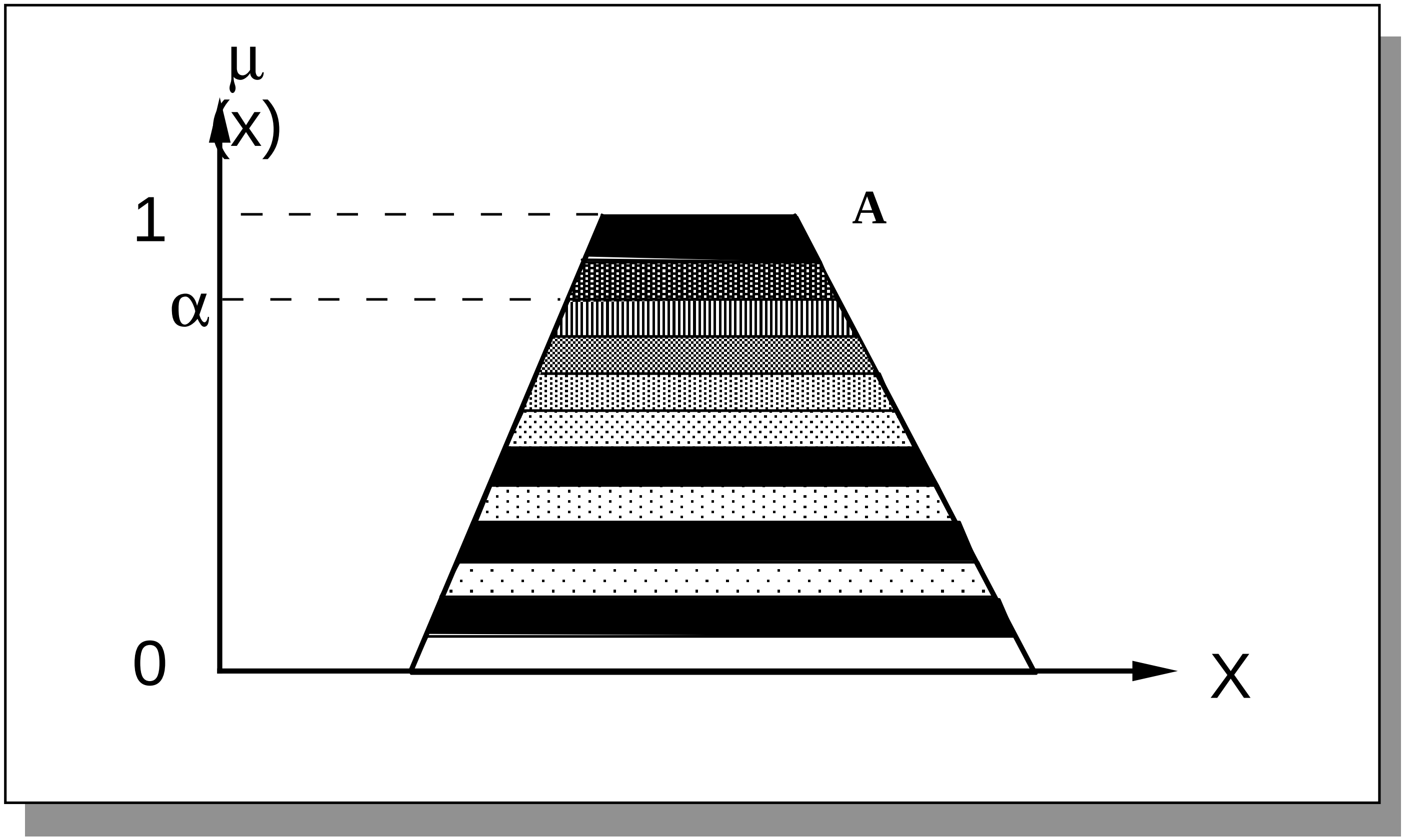
Support(A) $= \{ \mathbf{x} \mid A(x) > 0 \}$

Fuzzy Sets



$$\textbf{Bandwidth(A)} = \{x \mid A(x) \geq 0.5\}$$

Fuzzy Sets as Collections of Conventional Sets



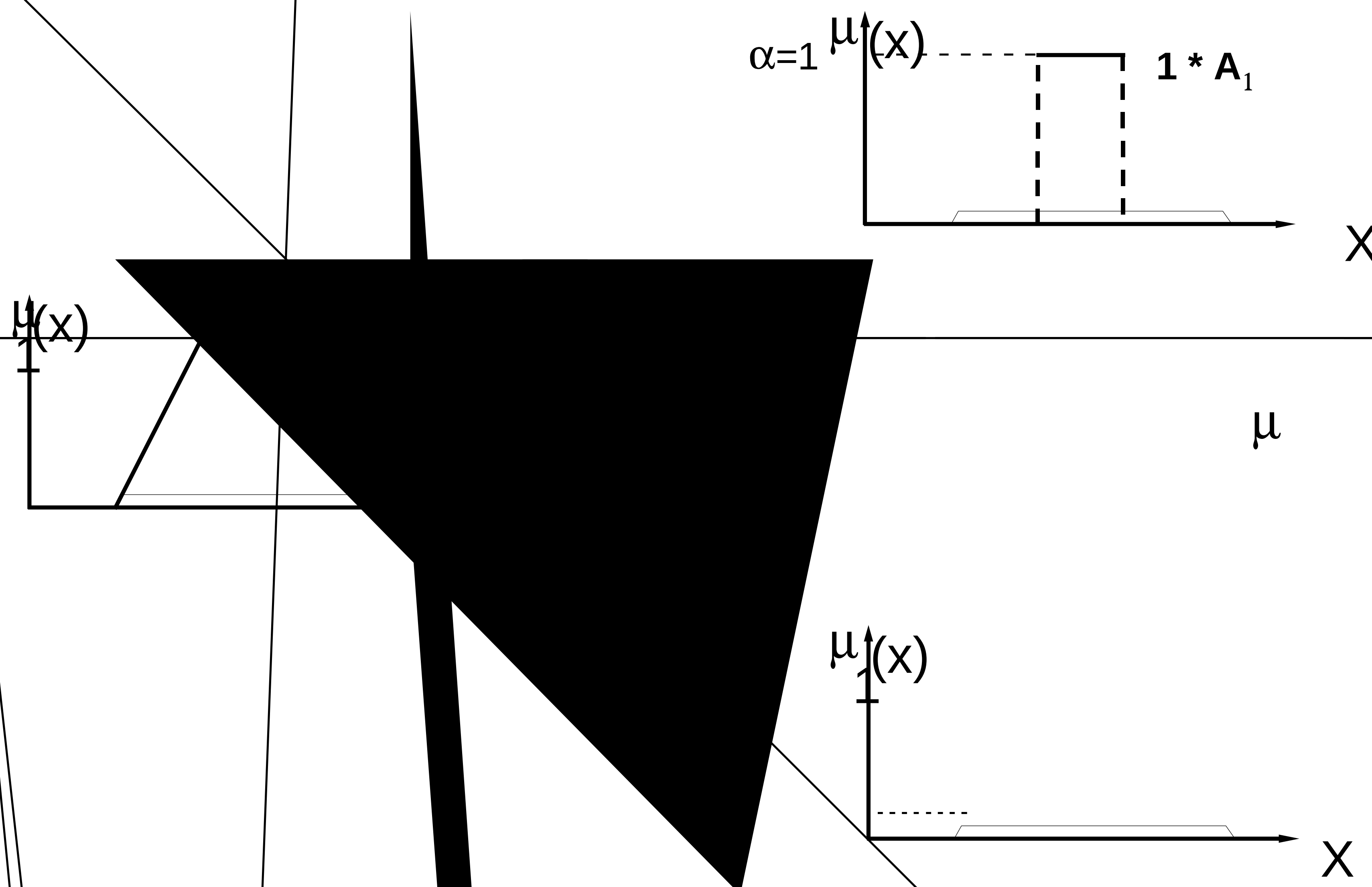
$$A_\alpha = \{x \mid A(x) \geq \alpha\},$$

Identity Principle (Decomposition Theorem)

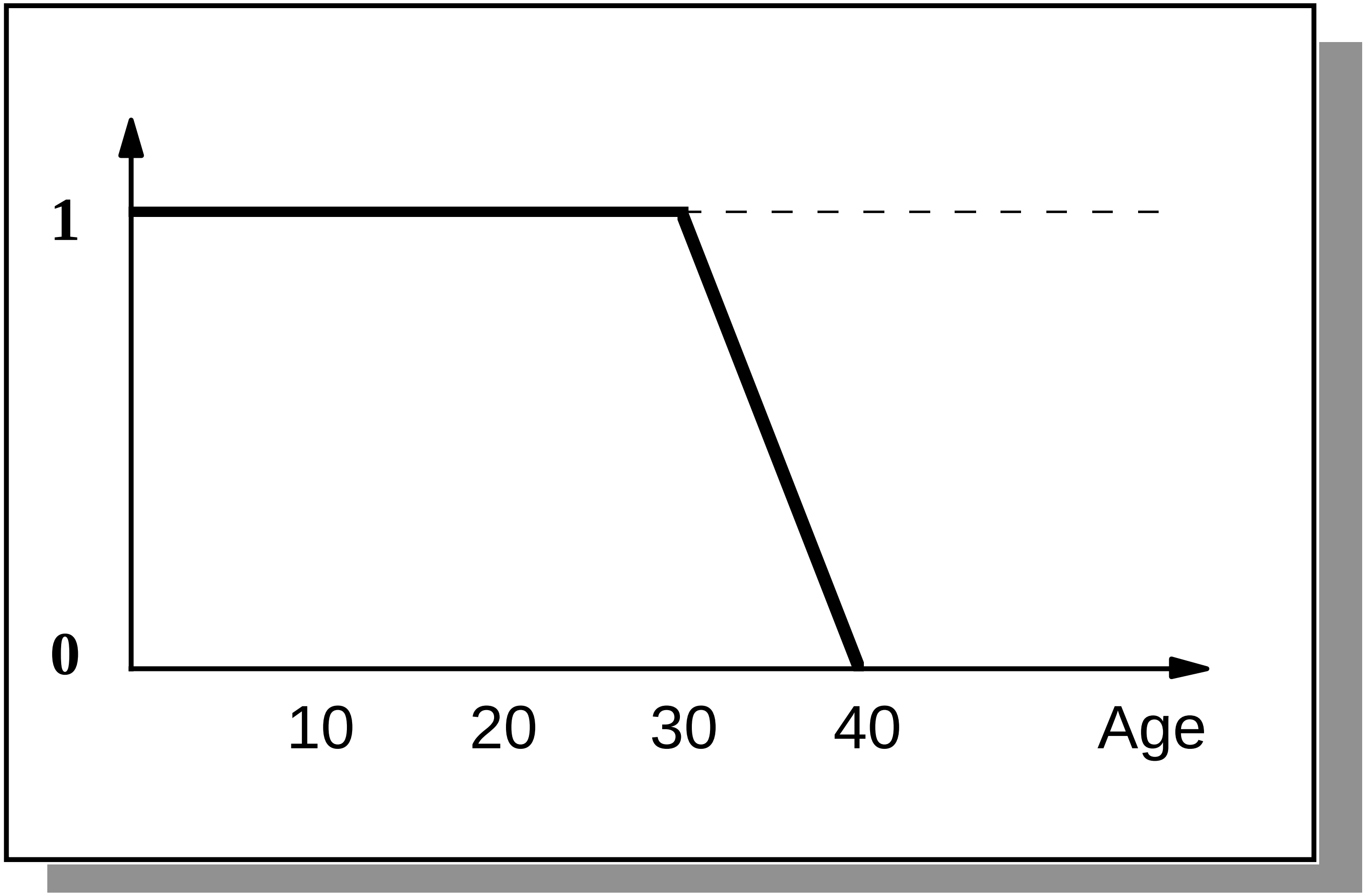
- **A fuzzy set A can be represented by the union of all its α-cut sets, weighted by their value α:**

$$A(x) = \bigcup_{\alpha \in [0,1]} \alpha \cdot A_{\alpha}(X)$$

Identity Principle (Decomposition Theorem)

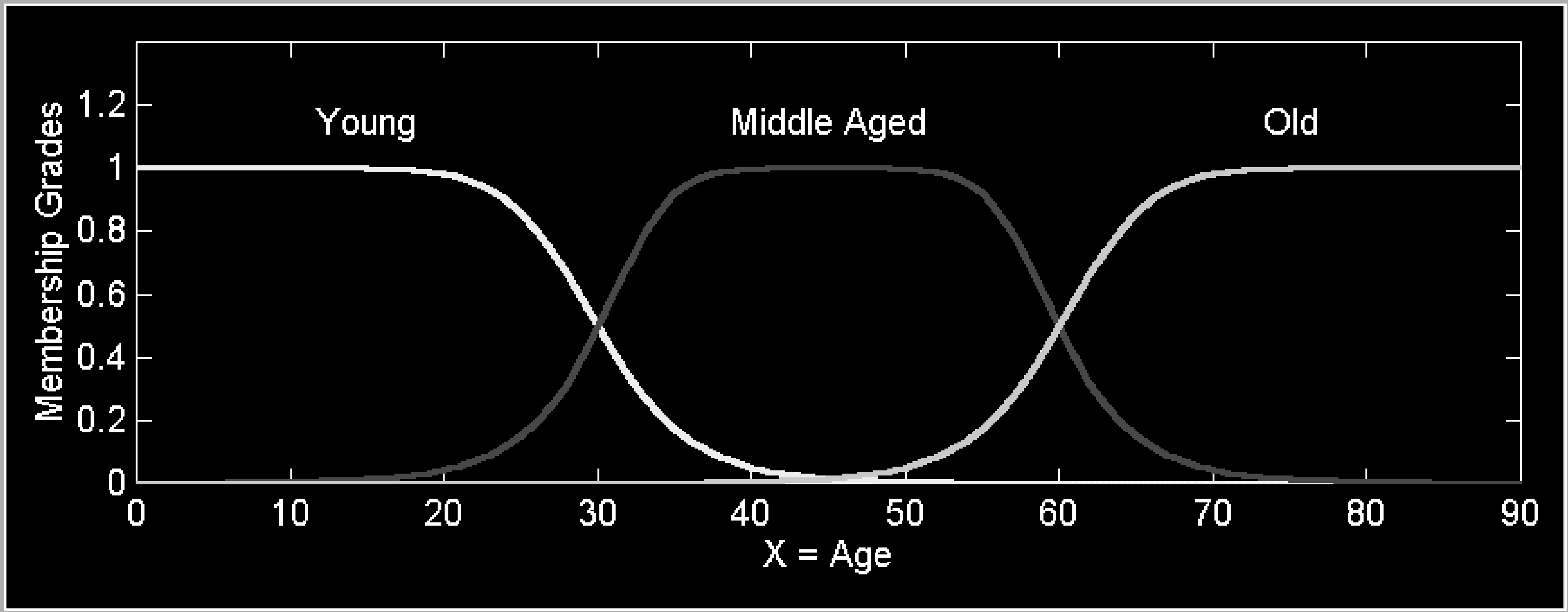


Young Persons



Fuzzy Partition

Fuzzy partitions formed by the linguistic values “young”, “middle aged”, and “old”:



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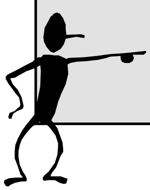
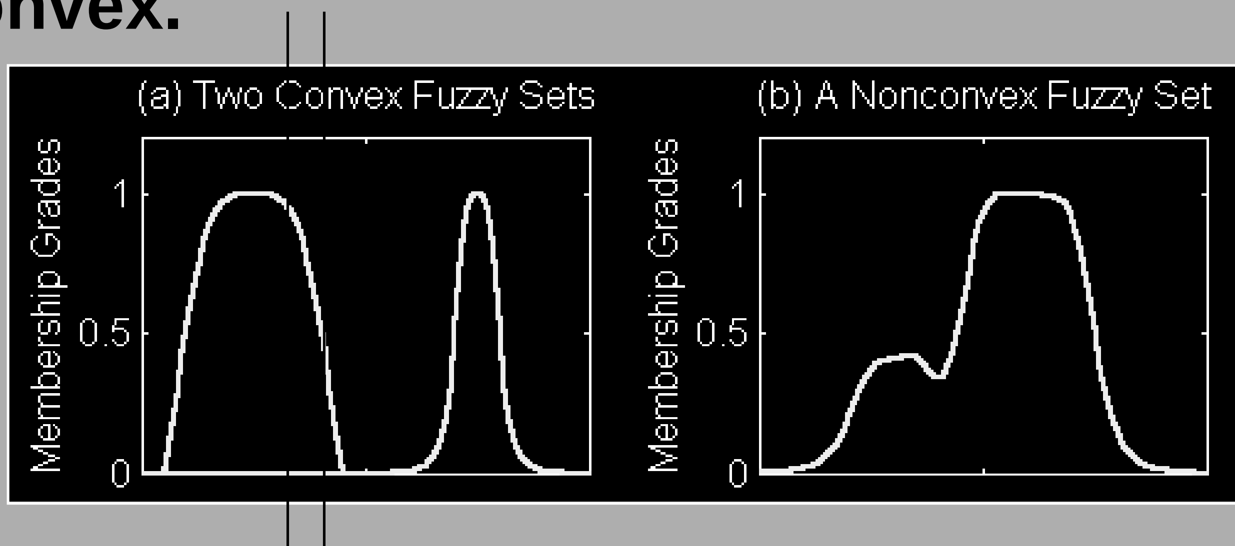


Convexity of Fuzzy Sets

A fuzzy set *A* is convex if for any λ in [0, 1],

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

Alternatively, *A* is convex is all its α-cuts are convex.



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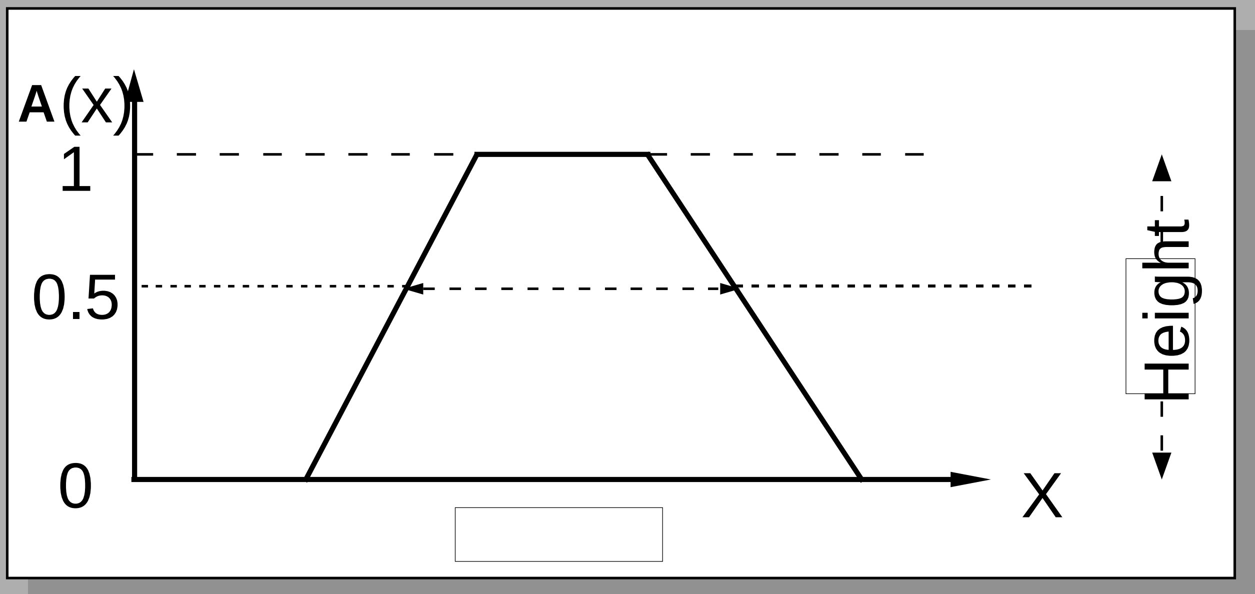
Normality of Fuzzy Sets

A fuzzy set *A* is *normal* if

$$Height(A) = Max_x A(x) = 1$$



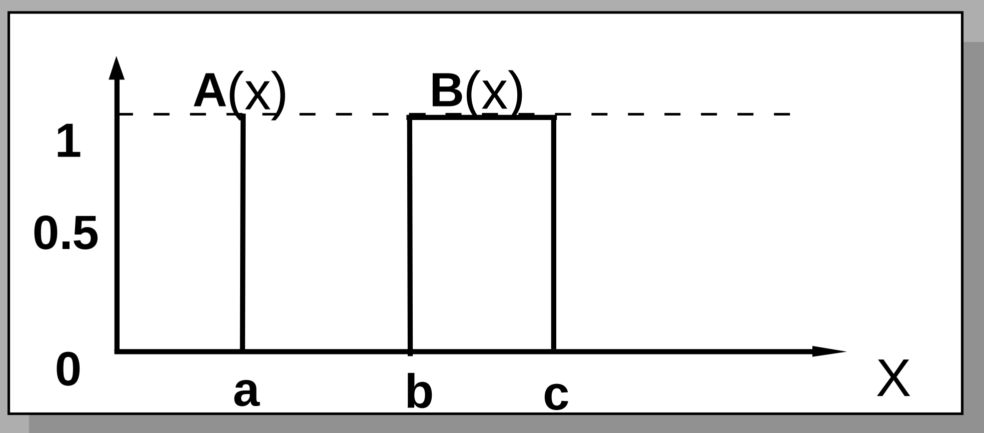
Fuzzy Set Representation of Crisp Numbers and Crisp Intervals

A crisp number *a* is represented by a fuzzy singleton

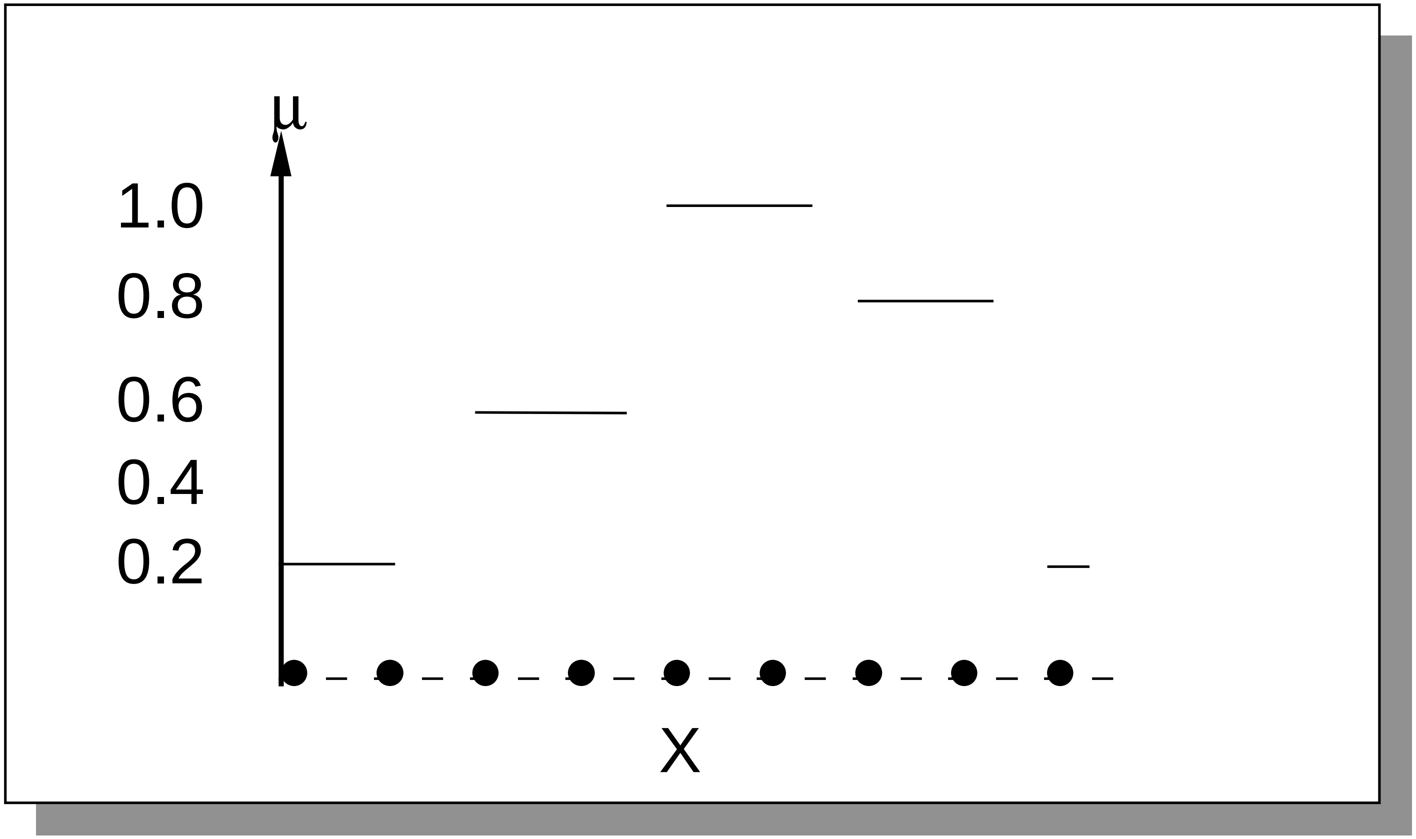
$$A(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

A crisp interval *[b,c]* is represented by a fuzzy set

$$B(x) = \begin{cases} 1 & \text{if } x \in [b,c] \\ 0 & \text{if } x \notin [b,c] \end{cases}$$

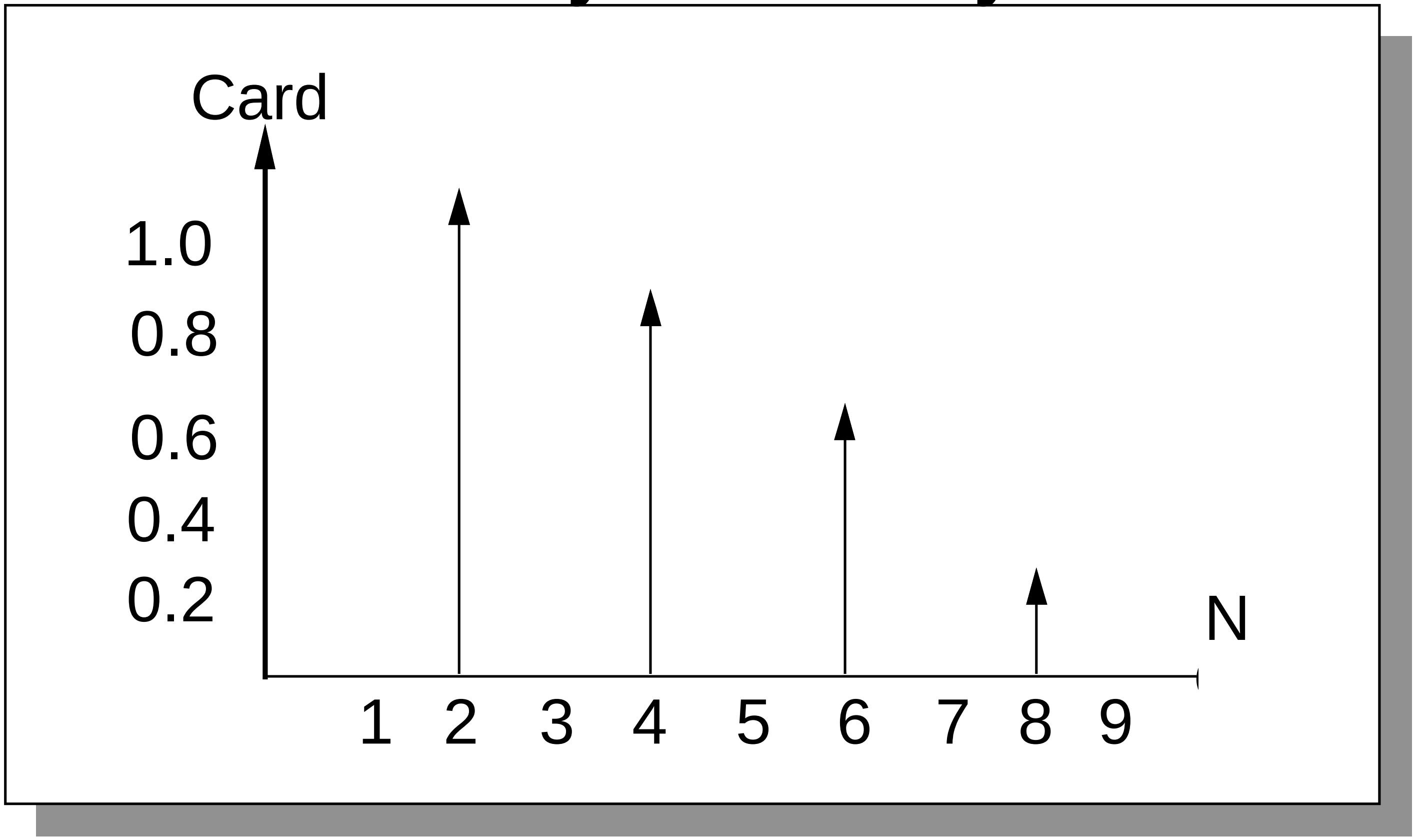


Scalar Cardinality



Σ Count (A) = Card(A) = 5.4

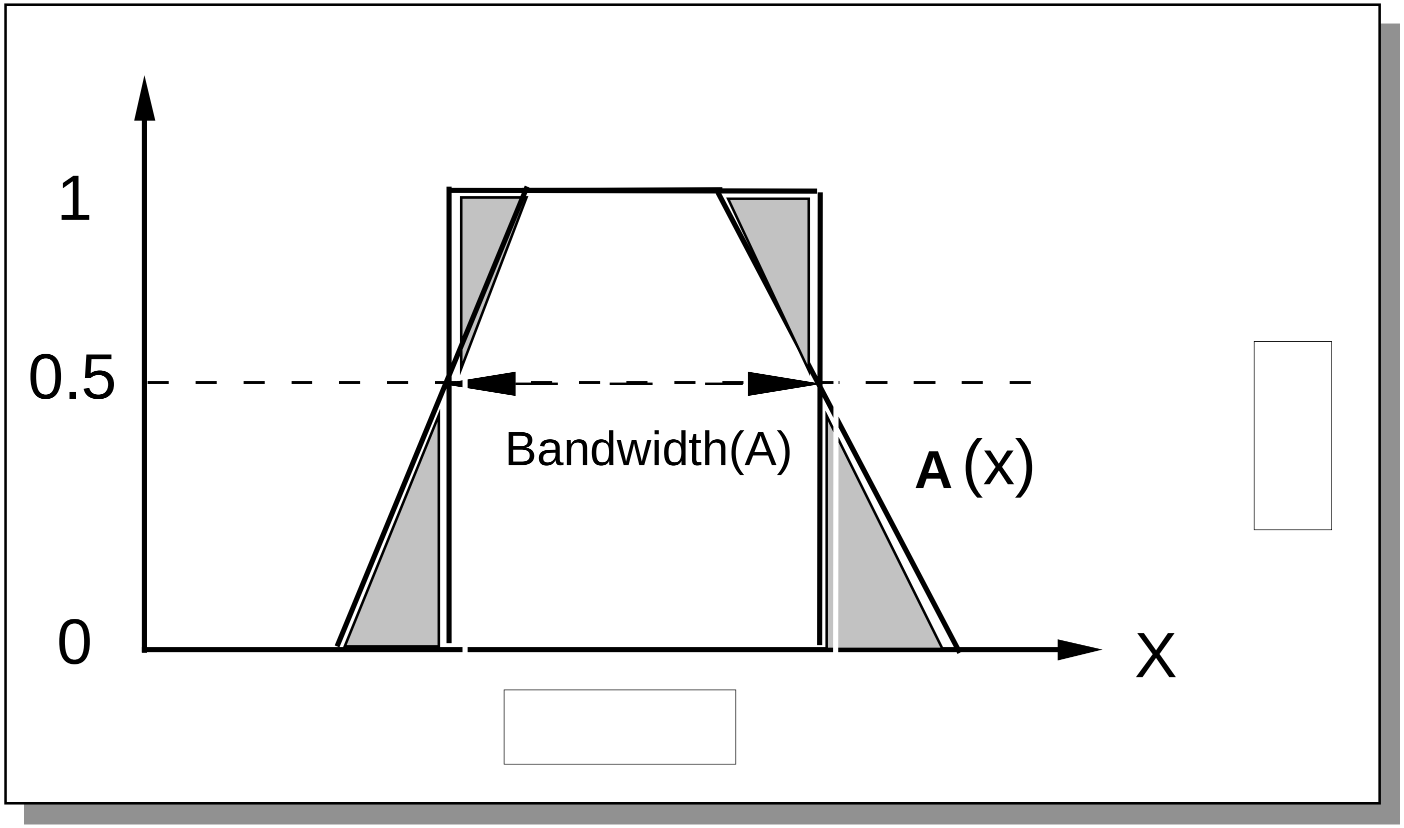
Fuzzy Cardinality



$$|A|(n) = \begin{cases} \alpha, & \text{if exists an } \alpha\text{-cut with } |A_\alpha| = n, \\ 0, & \text{otherwise.} \end{cases}$$

[The answer is a *fuzzy set* in the set of integer numbers]

Measure of Fuzziness



Measure of Fuzziness = Cardinality {|Bandwidth(A)- A(x)|}
= Cardinality { ▭ }

First Moment of a Fuzzy Sets

The First Moment of a discrete Fuzzy Set $A(x_i)$ is:

$$FirstMoment(A) = \frac{\sum_{i=1}^{n} A(x_i) * x_i}{\sum_{i=1}^{n} A(x_i)}$$

The First Moment of a continuous Fuzzy Set A(x) is:

$$FirstMoment(A) = \frac{\int A(x) * x\,dx}{\int A(x)\,dx}$$