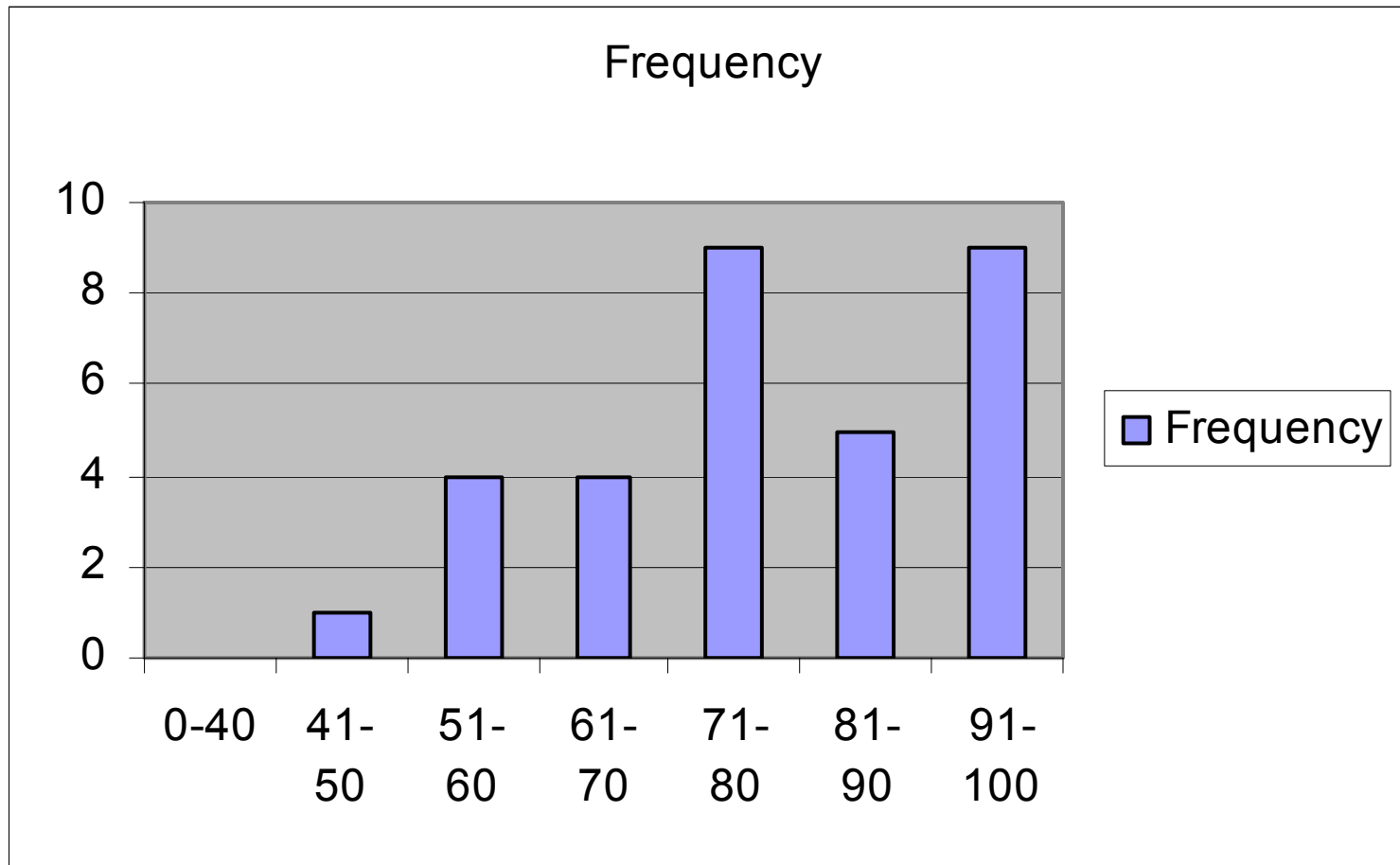


More Functions of Random Variables

Lecture 20

November 22

Quiz 2 results



Mean 78

Method of cdf

Basic idea: If $U=g(x)$, Find cdf of U as a function of cdf of X . Differentiate.

Order Statistics

Let X_1, X_2, \dots, X_n be a random sample i.e. x_i are independent, identically distributed (iid). The order statistics are

$$Y_1 = \min(X_1, \dots, X_n)$$

$$Y_n = \max(X_1, \dots, X_n)$$

$$Y_i = i^{\text{th}} \text{ smallest value of } X_1, \dots, X_n$$

Do maxworksheet

Maximum

Let X_1, X_2, \dots, X_n be a random sample from a continuous population with CDF $F(X)$, pdf $f(x)$. Find pdf of Y_n .

$$\text{Let } Y_n = \max(X_1, \dots, X_n)$$

$$Y_n \leq y_n \Rightarrow \forall i, X_i \leq y_n$$

$$F_{Y_n}(y_n) = P(Y_n \leq y_n)$$

$$= P(X_1 \leq y_n, \dots, X_N \leq y_n)$$

$$= P(X_1 \leq y_n) \dots P(X_N \leq y_n)$$

$$= P(X \leq y_n)^n = F_X(y_n)^n$$

Maximum continued

Find pdf from cdf

$$F_{Y_n}(y_n) = F_X(y_n)^n$$

$$f_{Y_n}(y_n) = \frac{\partial F_{Y_n}(y_n)}{\partial y_n} = nF_X(y_n)^{n-1} f_X(y_n)$$

Value set is the same as for X

Max example

Let the how far a car can drive on highway before driver is taken away by aliens. Let X_1, X_2, \dots, X_9 be how far nine cars make it, respectively. Assume the cars are independent and that how far they drive is uniform $[0, 200]$. What is the pdf of maximum distance the cars make it?

Solution

We know

$$f_X(x) = \frac{1}{200} \quad 0 < x < 200$$
$$F_X(x) = \begin{cases} 0 & x < 0 \\ x/200 & 0 \leq x < 200 \\ 1 & 200 \leq x \end{cases}$$

So

$$F_{Y_n}(y_n) = F_X(y_n)^9 = \left(\frac{y_n}{200}\right)^9$$

$$f_{Y_n}(y_n) = 9F_X(y_n)^8 f_X(y_n) = 9\left(\frac{y_n}{200}\right)^8 \frac{1}{200} \quad 0 < y_n < 200$$

Minimum

Do worksheet 2

Minimum

Let X_1, X_2, \dots, X_n be a random sample from a continuous population with CDF $F(X)$, pdf $f(x)$. Find pdf of Y_1 .

$$\text{Let } Y_1 = \min(X_1, \dots, X_n)$$

$$Y_1 > y_1 \Rightarrow \forall i, X_i > y_1$$

$$1 - F_{Y_1}(y_1) = P(Y_1 > y_1)$$

$$= P(X_1 > y_1, \dots, X_n > y_1)$$

$$= P(X_1 > y_1) \dots P(X_n > y_1)$$

$$= P(X > y_1)^n = (1 - P(X \leq y_1))^n = (1 - F_X(y_1))^n$$

Minimum continued

Find pdf from cdf

$$F_{Y_1}(y_1) = 1 - P(Y_1 > y_1) = 1 - (1 - F_X(y_1))^n$$

$$f_{Y_1}(y_1) = \frac{\partial F_{Y_1}(y_1)}{\partial y_1} = n(1 - F_X(y_1))^{n-1} f_X(y_1)$$

Value set is the same as for X

Min example

Let the how far a car can drive on highway before driver is taken away by aliens. Let X_1, X_2, \dots, X_9 be how far nine cars makes it, respectively. Assume the cars are independent and that how far they drive is uniform $[0, 200]$. How far will the first driver to be abducted make it?

Solution

We know

$$f_X(x) = \frac{1}{200} \quad 0 < x < 200$$
$$F_X(x) = \begin{cases} 0 & x < 0 \\ x/200 & 0 \leq x < 200 \\ 1 & 200 \leq x \end{cases}$$

So

$$F_{Y_1}(y_1) = (1 - F_X(y_1))^9 = \left(1 - \frac{y_1}{200}\right)^9$$

$$f_{Y_1}(y_1) = 9(1 - F_X(y_1))^8 f_X(y_1) = 9 \left(1 - \frac{y_1}{200}\right)^8 \frac{1}{200} \quad 0 < y_n < 200$$

General Order Statistics

In the book it shows that pdf of kth order statistic of a random sample with pdf $f(x)$ $a < x < b$ is

$$g_k(y_k) = \frac{n!}{(k-1)!(n-k)!} [F_X(y_k)]^{k-1} [1 - F_X(y_k)]^{n-k} f(y_k) \quad a < y_k < b$$

Example

You and for other people are bidding at a silent auction. The bids are independent and you assume that the other bids are uniform on $[100,200]$ dollars. How much must you bid to be at least 90% sure of winning the bid?

You will win if your bid is greater than the other 4 bids, i.e. if Y_4 is the maximum of the 4 bids and you bid c , you win if $Y_4 \leq c$. Thus we want to find c such that $P(Y_4 \leq c) = .9$.

Solution

What is the cdf of Y_4 ?

$$\begin{aligned}P(Y_4 \leq c) &= P(Y_1 \leq c, Y_2 \leq c, Y_3 \leq c, Y_4 \leq c) \\&= P(Y_1 \leq c)P(Y_2 \leq c)P(Y_3 \leq c)P(Y_4 \leq c) \\&= P(Y \leq c)^4 \\&= \left(\frac{c-100}{200-100} \right)^4 = \frac{(c-100)^4}{100^4}\end{aligned}$$

Now compute c

$$\begin{aligned}.9 &= P(Y \leq c)^4 = \frac{(c-100)^4}{100^4} \\c &= 100(.9)^{1/4} + 100 = \$197.41\end{aligned}$$

Joint Transforms

If X is a vector of continuous R.V. (x_1, x_2, \dots, x_n)

If $U=G(X)$ is 1 to 1 and the inverse function $x=g^{-1}(u)=h(u)$ is differentiable, then the pdf of U is given by

$$f_U(u) = f_X(h(u)) |J(u)|$$

Where $|J(u)|$ is the determinant of the Jacobian

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} \end{vmatrix}$$

Example

Let X and Y each be the size of a scoop of mashed potatoes. Assume each independent and each has distribution uniform $(0,1)$. What is the distribution of the amount of mash potatoes in two scoops, e.g. find pdf of $X+Y$?

$$U = X + Y$$

$$f_{XY}(x, y) = 1 \quad 0 < X < 1 \quad 0 < Y < 1$$

Step 1: find 1 to 1 transform

$$U=X+Y \quad 0<U<2$$

$$V=Y$$

continued

- Step 2 – Solve for (x,y) in terms of (u,v)

$$U=X+Y$$

$$V=Y$$

$$Y=V \quad X=U-V$$

- Step 3 Calculate the range of U and V

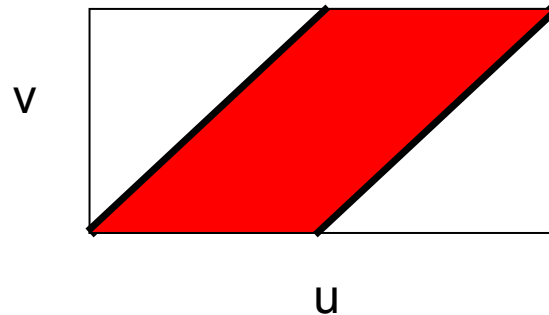
$$0 < Y < 1 \quad 0 < V < 1$$

$$0 < X < 1 \quad 0 < U - V < 1$$

Range of V depends on U

If $0 < U < 1$ then $0 < V < U$

If $1 < U < 2$ then $U - 1 < V < 1$



- Step 4 – Calculate Jacobian of
 $Y=V$ $X=U-V$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = (1)(1) - (0)(-1) = 1$$

Step 5 and 6

- Use formula to calculate the joint of U and V

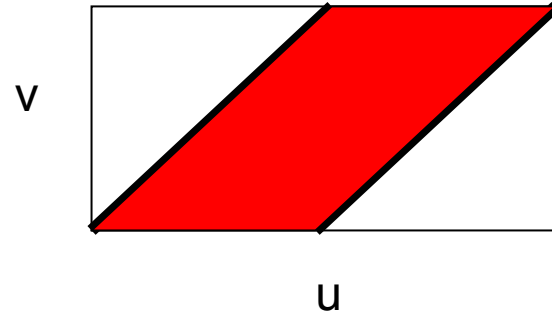
$$f_{U,V}(u,v) = f_{X,Y}(u-v,v) |J| = 1(1) \quad 0 < V < U, U-1 < V < 1$$

- Calculate marginal of U

$$f_U(u) = \int_R f(u,v) dv$$

Step 5 and 6

Calculate marginal of U



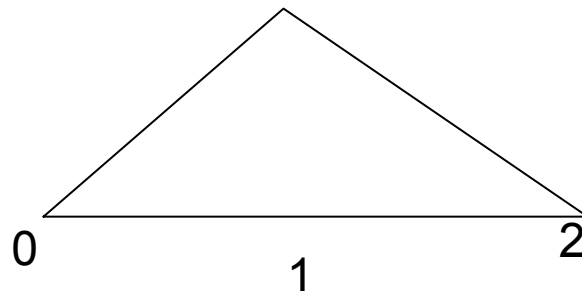
If $0 < u \leq 1$ then

$$f_U(u) = \int_0^u 1 dv = u \quad 0 < u \leq 1$$

If $1 < u < 2$

$$f_U(u) = \int_{u-1}^1 1 dv = 1 - (u - 1) \quad 1 < u < 2$$

Pdf is



Final pdf

$$f_U(u) = \begin{cases} u & 0 < u \leq 1 \\ 2 - u & 1 < u < 2 \\ 0 & \text{o.w} \end{cases}$$

- Step 2 – Solve for (x,y) in terms of (u,v)

$$U=X+Y$$

$$V=Y$$

$$Y=V \quad X=U-V$$

- Step 3 Calculate the range of U and V

$$0 < X < \infty \rightarrow 0 < U - V < \infty$$

$$0 < Y < \infty \rightarrow 0 < V < \infty$$

$$0 < V < U < \infty$$

Example

Let X and Y each be the weight of your turkey and potatoes, respectively. For convenience assume that X and Y have been normalized to be normal 0 and 1 and are independent. Your plate will balance if the ratio of X and Y is even. Assume each independent. What is the distribution of this ratio?

$$U = X / Y$$

$$f_{XY}(x, y) = \frac{1}{2\Pi} e^{-1/2(x^2+y^2)} \quad -\infty < X < \infty \quad -\infty < Y < \infty$$

Step 1: find 1 to 1 transform

$$U=X/Y$$

$$V=Y$$

continued

- Step 2 – Solve for (x,y) in terms of (u,v)

$$U=X/Y$$

$$V=Y$$

$$Y=V \quad X=UV$$

- Step 3 Calculate the range of U and V

$$-\infty < X < \infty \quad -\infty < Y < \infty$$

$$-\infty < U < \infty \quad -\infty < V < \infty$$

- Step 4 – Calculate Jacobian of

$$Y=V \quad X=UV$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} V & U \\ 0 & 1 \end{vmatrix} = |V - (0)| = |V|$$

Step 5

- Use formula to calculate the joint of U and V

$$f_{U,V}(u,v) = f_X(uv,v) |J|$$

$$= \frac{1}{2\Pi} e^{-1/2(u^2v^2+v^2)} |v| \quad -\infty < u < \infty$$

$$-\infty < v < \infty$$

- Calculate marginal of U

$$f_U(u) = \int_R f(u,v) dv$$

Step 6

Calculate marginal of U

$$\begin{aligned} f_U(u) &= 2 \frac{1}{2\Pi} \int_0^{\infty} e^{-v^2/2(1+u^2)} v dv \\ &= \frac{1}{\Pi} \int_0^{\infty} e^{-x(1+u^2)} dx \quad \text{let } v^2/2 = x \quad v dv = dx \\ &= \frac{1}{\Pi} \frac{1}{1+u^2} \quad -\infty < u < \infty \end{aligned}$$