Lecture 3: Combinatorics

Probability Theory and Applications

Fall 2005

September 6

Sample Space Method

- Define Experiment
- Define *S*, union of disjoint events E_i
- Assign reasonable probabilities to E_i
- Define Even *A* as collection of sample points

$$P(A) = \sum_{E_i \in A} P(E_i)$$

- Worksheet 3

Combinatorics

- Special Case: *Ei* equally likely then

$$P(A) = \frac{|A|}{|S|}$$

|*A*| = cardinality of *A*

- *So reduces to counting*
- *Counting Rules*

Multiplication Rule

- Experiment:
  - First part has *m* outcomes
  - Second part has *n* outcomes
    independent of first part

  Experiment has *m**n* sample points

Roll Three Fair Dice

|Sample Space| =

P(Roll 361) =

P(Roll 222) =

P(Roll three of a kind) =

P(Roll at least one 6) =

Permutation Rule

- An ordered arrangement of r distinct objects
- The number of ways of ordering r objects taken from n distinct objects
- Denote

$$P_r^n = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

- Number of ways 20 people can have 20 different birthdays?

$$P_{20}^{365} = 365(364)(363)\ldots(346) = \frac{365!}{(365-20)!}$$

Birthday Problem

What is probability 2 people have same birthdays?

What is probability 20 people have 20 different birthdays?

What is the probability at least 2 people in 20 have same birthday?

# of people	P(all have different birthdays)
10	.883
20	.589
30	.294
40	.1
50	.030
…	…
100	$.3 \times 10^{-7}$

Correct or not?

From the May 2005 RSS News:

- He tried his best--but in the end newborn Casey-James May missed out on a 48 million-to-one record by four minutes. His father Sean, grandfather Dered and great-grandfather Alistair were all born on the same date - March 2. But Casey-James was delivered at 12.04 am on March 3....

Combinations

Choices without regard to order

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

# of unordered samples of size *r* taken from *n* distinct objects

Poker (5 card stud)

Number of possible poker hands when playing 5 card stud

$$\binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

P(Heart Royal Flush)=

P(Royal Flush)=

Example

- You have a box with 5 defectives and 3 goods. Draw 3 items **without replacement**.
- P(DDG)
- P(exactly two defectives)

Sampling without replacement

k defectives

n – k goods

n items

→ draw r items **without** replacement

P(x defectives and r-x goods)=

$$= \frac{\binom{k}{x}\binom{n-k}{r-x}}{\binom{n}{r}}$$

Example

- You have a box with 5 defectives and 3 goods. Draw 3 items **<u>with replacement</u>**.

- P(DDG)

- P(exactly two defectives)

Binomial Distribution

In general for n draws with replacement

P(x defectives,n-x goods)=

$$\binom{n}{x} p^x (1-p)^{n-x}$$

Partitions

- The number of ways of partitioning *n* *distinct* objects into *k* distinct groups containing $n_1, n_{2,} \ldots, n_k$ objects respectively.

$$\binom{n}{n_1}\binom{n-n_1}{n-n_2}\cdots\binom{n_k}{n_k} = \frac{n!}{n_1!\,n_2!\ldots n_k!}$$

How many different solitaire games?

Piles of cards 1 2 3 4 5 6 7 + 24

$$\frac{52!}{1!2!3!4!5!6!7!24!} =\sim 10^{33}$$

Example

In a restaurant, 8 customers order dinner. They order 3 Chickens, 4 Steaks, and 1 Eggplant surprise. The waiter forgets who ordered what. What’s the probability that he will guess correctly who ordered what?

Worksheet 4

In Yahtzee, five fair dice are rolled. What is the probability of each of the following results?

- Four of a kind (four of a kind and one singleton) , e.g. 5 5 5 5 2
- Full house (two and three of a kind), e.g. 2 2 3 3 3
- Three of a kind (three of a kind and two more distinct rolls) e.g. 2 4 2 2 1
- Two pairs (two distinct pairs and a singleton), e.g. 2 2 3 3 1