PROBABILITY THEORY AND APPLICATIONS
MATP 4600 and DSES 4750
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LECTURE 1
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PROBLEM. (URN PROBLEM) An urn contains 11 red balls and 3 white balls. Two balls are drawn simultaneously. How likely is it that both balls are red?

> We have no tools for determining the correctness of an answer to this question. However, a fraction resulting from computations done in a mathematical model (probability model) of the problem can be determined to be correct, or not. If, within the model, the fraction is determined to be correct, it can be lifted out of the model and used as a "real world" answer to the question. So, questions of the form "How likely is it..?", or "What is the probability of ...?" need to be interpreted as "Within a reasonable mathematical model, compute the probability of ...." We illustrate this thinking in the analysis of the urn problem given next.

MATHEMATICAL MODEL OF URN PROBLEM

Model each of the balls with a positive integer: 1-11 for the red balls and 12-14 for the white balls. (Note that this act of numbering builds into the model a way of distinguishing balls not present in the original problem statement.)

Let S denote the set of all pairs of distinct integers, $\{i, j\}$, $1 \leq i, j \leq 14$, $i \neq j$. The set S models all possible draws.

> The set S is called the *sample space*. The elements of 2^S are called *events*. These two notions will be formalized after discussion of the urn problem.

Note that the number of elements in S is

$$\binom{14}{2} = \frac{14!}{12!\,2!} = \frac{(14)(13)}{2} = 91.$$

Assign to each element of S the probability,

$$\frac{1}{91}.$$

This is the key modeling step.[1]

Let E denote the event (subset of S) that models all pairs of red balls.

$$E = \{\{i, j\} : 1 \le i, j \le 11,\, i \ne j\}.$$

Note that the number of elements in E is

$$\binom{11}{2} = \frac{11!}{9!\,2!} = \frac{(11)(10)}{2} = 55.$$

Next we *compute* the probability of the the set, E, denoted here by $P(E)$, using the fact that the probability of a (finite) set is the sum of the probabilities of its elements. (This idea will be formalized below.)

$$P(E) = \sum_{s \in E} P(s) = \sum_{s \in E} \frac{1}{91} = \frac{55}{91}.$$

(We note that it would be more correct to write $P(\{s\})$ instead of $P(s)$, but find that a notation burden. The correctness of the notation, $P(\{s\})$, will become apparent below.)

Finally we "lift out of the model" the fraction $\frac{55}{91}$ and use it as a "real world" answer to the question "How likely is it that both balls are red?"

What just happened? We computed the probability of an event within a probability model. In the process we used the notion of a probability function without first detailing its properties. That is done next in a formal way.

[1]In general, almost all debate on the reasonableness of a probability model, centers on the assignment of probabilities. Such assignment is sometimes based on experimental data, sometimes on human intuition, and sometimes a "wild guess".

DEFINITION. (PROBABILITY FUNCTION) Let S denote a non-empty finite or countably infinite set and let 2^S denote the set of all subsets of S (the power set of S). A real-valued function P defined on 2^S is a **probability function** if

- $P(E) \geq 0, \forall E \in 2^S$,
- $P(S) = 1$,
- $P(E_1 \cup E_2) = P(E_1) + P(E_2)$, $\forall E_1, E_2 \in 2^S$, $E_1 \cap E_2 = \emptyset$.

DEFINITION. (SAMPLE SPACE) The set S is called a sample space.[2]

DEFINITION. (EVENTS) The elements of 2^S are called **events** and the elements of S are called *elementary events*.

Note. If the set S is uncountably infinite, the domain of P needs to be changed. In this case it is assumed that P is defined on some but not all of the subsets of S. A further discussion of this is delayed, since our initial investigations center on finite or countably infinite sets S.

In the urn example the probability function, P, is indeed defined on all of 2^S (all subsets of S). For $E \subseteq S$, the value of $P(E)$ is $\frac{1}{91}$ times the number of elements in E. This observation follows from the third bullet in the definition of a probability function. In the next example we compute the probability of another event in the above probability model.

PROBLEM. (ANOTHER URN PROBLEM) An urn contains 11 red balls and 3 white balls. Two balls are drawn simultaneously. How likely is it that one ball is red and the other is white?

MATHEMATICAL MODEL OF URN PROBLEM

Let S denote the set of all pairs of distinct integers, $\{i, j\}$, $1 \leq i, j \leq 14$, $i \neq j$. The set S models all possible draws; it is the sample space for our model.

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[2]This definition may leave the reader hanging. The intent is that any non-empty set is a sample space if a probability function is defined on its power set.

Note that the number of elements in S is

$$\binom{14}{2} = \frac{14!}{12!\,2!} = \frac{(14)(13)}{2} = 91.$$

Define a probability function, P, on 2^S by the rule

$$P(s) = \frac{1}{91},\ s \in S.$$

Note that the probability of every subset of S, is found by adding the the sum of the probabilities of its elements (using the third bullet in the definition of a probability function). So P is indeed defined on 2^S, not just S.

Let E denote the event (subset of S) that models all pairs corresponding to exactly one red ball.

$$E = \{\{i, j\} : 1 \leq i \leq 11,\ 12 \leq j \leq 14\}.$$

Note that the number of elements in E is

$$(11)(3) = 33.$$

Next we find

$$P(E) = \sum_{s \in E} P(s) = \sum_{s \in E} \frac{1}{91} = \frac{33}{91}.$$

Finally we "lift out of the model" the fraction $\frac{33}{91}$ and use it as a "real world" answer to the question "How likely is it that one ball is red and the other is white?"

In the above problems it was postulated that the two balls are drawn from the urn simultaneously. What if the balls are chosen one-at-a-time, e.g., the first ball is drawn and set on a table and then the second ball is drawn from the remaining 13 balls? How likely is it that one ball is red and the other is white?

There are at least two good ways to think about this problem.

> (i) One can think of drawing two balls simultaneously by using two hands and picking up a ball with each. One can think of the left hand choosing one of 14 balls and the right hand choosing

any of the 14 balls that the left hand has not chosen. So if the left hand lifts a ball a fraction of a second before the right hand, the balls are not chosen simultaneously but one-at-a-time. One would not want a slight lack of hand coordination to effect the mathematical model; one wants the model for the "simultaneous selection" to yield the same answer as a model for the "one-at-a-time" selection. Having made this observation one can declare that the model for the "one-at-a-time" selection is exactly the same as the model for the "simultaneous selection". (This declaration is part of the modeling process.)

(ii) Let S denote the set of all ordered pairs (i, j), $1 \leq i, j \leq 14$, $i \neq j$. The set S models all possible one-at-a-time draws. It is the sample space of our model.

Note that the number of elements in S is

$$(14)(13) = 182.$$

Define a probability function, P, on 2^S by the rule

$$P(s) = \frac{1}{182},\ s \in S.$$

Note that the probability of every subset of S, is found by adding the sum of the probabilities of its elements (using the third bullet in the definition of a probability function). So, P is indeed defined on 2^S and not just S.

Let E denote the event (subset of S) that models all ordered pairs corresponding to one red ball and one white ball.

$E = (\{i, j) : 1 \leq i \leq 11,\ 12 \leq j \leq 14,\ \text{or}\ 12 \leq i \leq 14,\ 1 \leq j \leq 11\}.$

Note that the number of elements in E is

$$(11)(3) + (3)(11) = 66.$$

Next we find

$$P(E) = \sum_{s \in E} P(s) = \sum_{s \in E} \frac{1}{182} = \frac{66}{182} = \frac{33}{91}.$$

Finally we "lift out of the model" the fraction $\frac{33}{91}$ and use it as a "real world" answer to the question "How likely is it that one ball is red and the other is white."