Nonlinear Inequality Constraints Example

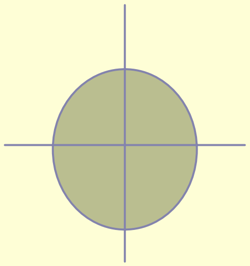
Example

$$\max x_1$$

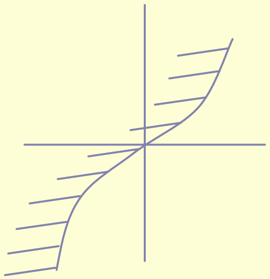
$$\text{s.t.} \quad x_1^{\ 2} + x_2^{\ 2} \leq 1$$

$$(x_1 - 1)^3 - x_2 \leq 0$$

$x_1^{\ 2} + x_2^{\ 2} \leq 1$



$(x_1 - 1)^3 - x_2 \leq 0$



Convert to standard form

$$\begin{aligned} \min \quad & -x_1 \\ \text{s.t.} \quad & -x_1^{\,2} - x_2^{\,2} + 1 \geq 0 \\ & -(x_1 - 1)^3 + x_2 \geq 0 \end{aligned}$$

$$x^* = [1,0] \qquad \lambda^* = \left[\frac{1}{2},0\right]$$

Primal feasibility satisfied

Both constraints active

Dual Feasibility: $L(x,\lambda) = -x_1 - \lambda_1(-x_1^{\,2} - x_2^{\,2} - 1) - \lambda_2(-(x_1 - 1)^3 + x_2)$

Check FONC

$$\nabla_x L(x,\lambda) = \overbrace{\begin{bmatrix}-1\\0\end{bmatrix}}^{\nabla f(x)} - \lambda_1\begin{bmatrix}-2x_1\\-2x_2\end{bmatrix} - \lambda_2\begin{bmatrix}-3(x_1-1)^2\\1\end{bmatrix}$$

$$x^* = [1,0] \qquad \lambda^* = \left[\frac{1}{2},0\right]$$

$$\frac{\partial L(x^*,\lambda^*)}{\partial x_1} = -1 + 2(½)(1) + (3)(0)(1-1)^2 = 0$$

$$\frac{\partial L(x^*,\lambda^*)}{\partial x_2} = 0 + 2(½)(0) - 0 = 0$$

$$\lambda^* \geq 0$$

Solution solves KKT cond

Assuming two constraints active

$$\frac{\partial L}{\partial x_1} = -1 + 2\lambda_1 x_1 + 3\lambda_2(x_1-1)^2 = 0$$

$$\frac{\partial L}{\partial x_2} = 0 + 2\lambda_1 x_2 - \lambda_2 = 0$$

$$-x_1^2 - x_2^2 + 1 = 0$$

$$-(x_1-1)^3 + x_2 = 0$$

$$\lambda_1, \lambda_2 \geq 0$$

Complementarity

Complementarity: $\lambda_i g_i(x) = 0$

$$\left.\begin{array}{l} g_1(x) = 0 \\ g_2(x) = 0 \end{array}\right\} \text{thus complementarity holds}$$

For inactive constraints, $\lambda_i = 0$

More FONC, SONC

Let's check necessary conditions

Is x* regular, i.e. are the gradients of the active constraints linearly independent?

$$\begin{bmatrix} \nabla g_1(x)^T \\ \nabla g_2(x)^T \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

So x* is regular, and x* is a KKT point, so FONC are satisfied. The null space is empty so

SONC: $Z^T \nabla_{xx}^2 L(x,\lambda) Z$ p.s.d. is vacuously satisfied.

A_+, Z_+, etc.

$$A_+ = \begin{bmatrix} -2x_1 & -2x_2 \end{bmatrix}$$

Only first constraint is active
And non-degenerate (positive multiplier)

$$= \begin{bmatrix} -2 & 0 \end{bmatrix}$$

$$Z_+ = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\nabla_x L(x,\lambda) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} - \lambda_1 \begin{bmatrix} -2x_1 \\ -2x_2 \end{bmatrix} - \lambda_2 \begin{bmatrix} -3(x_1-1)^2 \\ 1 \end{bmatrix}$$

$$\nabla_{xx}{}^2 L(x,\lambda) = \begin{bmatrix} 2\lambda_1 + 6\lambda_2(x_1-1) & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Conclusion

Just showed that x*, λ* is a KKT point.

Let's check SOSC: $Z_+^T \nabla_{xx}^2 L(x,\lambda) Z_+$ is positive definite

The general Jacobian of this problem for the active constraints:

$$\begin{bmatrix} \nabla g_1(x)^T \\ \nabla g_2(x)^T \end{bmatrix} = \begin{bmatrix} -2x_1 & -2x_2 \\ -3(x_1-1)^2 & 1 \end{bmatrix}$$

We only need non-degenerate active constraints

$$Z_+^T \nabla_{xx}^2 L(x,\lambda) Z_+ = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 > 0$$

so SOSC satisfied; x* is a strict local minimizer