Solutions

Physics PhD Qualifying Examination
Part I – Wednesday, August 20, 2008

Name:__________________________

(please print)

Identification Number:__________

STUDENT: Designate the problem numbers that you are handing in for grading in the
appropriate left hand boxes below. Initial the right hand box.

PROCTOR: Check off the right hand boxes corresponding to the problems received from
each student. Initial in the right hand box.

|   |   |
|---|---|---|
| 1 |   | Student’s initials |
| 2 |   | # problems handed in: |
| 3 |   | Proctor’s initials |
| 4 |   |   |
| 5 |   |   |
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| 10|   |   |

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE
   COLLATED AND GRADED BY THE ID NUMBER ABOVE.

2. Use at least one separate preprinted answer sheet for each problem. Write on only one
   side of each answer sheet.

3. Write your identification number listed above, in the appropriate box on each preprinted
   answer sheet.

4. Write the problem number in the appropriate box of each preprinted answer sheet. If
   you use more than one page for an answer, then number the answer sheets with both
   problem number and page (e.g. Problem 9 – Page 1 of 3).

5. Staple together all the pages pertaining to a given problem. Use a paper clip to group
together all eight problems that you are handing in.

6. Hand in a total of eight problems. A passing distribution will normally include at least
   three passed problems from problems 1-5 (Mechanics) and three problems from problems
   6-10 (Electricity and Magnetism). DO NOT HAND IN MORE THAN EIGHT
   PROBLEMS.

7. YOU MUST SHOW ALL YOUR WORK.
A particle slides freely and without friction on the top surface of a spherically-shaped object with radius $R$. The mass of the particle is $m$ and the magnitude of the gravitational acceleration is $g$. The particle is initially at the top of the sphere with infinitesimally small velocity. Determine the angle $\theta$ at which the particle "takes off", i.e., the angle at which the particle separates from the surface of the sphere. See illustration below.

A plane pendulum consists of a mass $m$ suspended by a massless spring with unextended length $b$ and spring constant $k$. Find Lagrange's equations of motion.
Two masses \( m_1 \) and \( m_2 \) \((m_1 = m_2 = M)\) are connected to each other by a spring with spring constant \( \kappa_{12} \) and to fixed points at the two ends by springs with spring constant \( \kappa \), as shown below.

(1) Write down the one-dimensional equations of motion for \( m_1 \) and \( m_2 \).

(2) Assuming that the motion of the masses is oscillatory (i.e., \( x_j(t) = B_j e^{i \omega t}, \ j = 1, 2 \)), simplify the equations of motion; solve this pair of simultaneous equations; and derive the characteristic frequencies (or normal frequencies) for the system.

(3) Given the schematic shown below, associate the appropriate normal modes to the derived characteristic frequencies. Explain your answer (e.g. in terms of either “phase”-argument or “energy”-argument.)
Two masses $m_1$ and $m_2$ are initially separated by a distance $r_o$ and are released from rest. Assume that the only force acting between the two masses is the gravitational force. Calculate the speed $v_1$ and $v_2$ of the two masses as a function of the instantaneous separation $r$, the initial separation $r_o$, $m_1$, $m_2$, and $G$ (gravitational force constant).

An atom in its ground state has mass $m$. It is initially at rest, in an excited state of excitation energy $\Delta \varepsilon$. It then makes a transition to the ground state by emitting one photon. Find the frequency of the photon, taking into account the relativistic recoil of the atom. Express your answer also in terms of the mass $M$ of the excited atom.

Two spherical cavities of radii $a$ and $b$ are hollowed out from the interior of a (neutral) conducting sphere of radius $R$. At the center of each cavity a point charge is placed -- call these charges $q_a$ and $q_b$ (see illustration below).

(a) Find the surface charges $\sigma_a$, $\sigma_b$, and $\sigma_R$.
   [$\sigma_a$ is the surface charge on the surface of cavity $a$, $\sigma_b$ is the surface charge on the surface of cavity $b$, and $\sigma_R$ is the surface charge on the conducting sphere.]
(b) What is the electric field outside the conducting sphere?
(c) What is the electric field inside each cavity?
(d) What is the force on $q_a$ and $q_b$?
(1) Write down
   (i) the expression for the electric field $\mathbf{E}$ in terms of the scalar potential $V$ and the vector potential $\mathbf{A}$;
   (ii) the relation between the magnetic field $\mathbf{B}$ and the vector potential $\mathbf{A}$.

(2) Given the scalar potential $V = 0$ and vector potential $\mathbf{A}$ below,

\[
\mathbf{A} = \begin{cases} 
\frac{\mu_0 \alpha}{4c} (ct - |x|)^2 \mathbf{\hat{z}} & \text{for } |x| < ct \\
0 & \text{for } |x| > ct
\end{cases}
\]

find $\mathbf{E}$ and $\mathbf{B}$.

(3) Write down
   (i) the expression that relates the discontinuity in the electric displacement vector $\mathbf{D}$ to the free surface charge density $\sigma_f$ at the $x = 0$ interface.
   (ii) the expression that relates the discontinuity in magnetic field $\mathbf{B}$ to the free surface current density $\mathbf{K}_f$ at the $x = 0$ interface.

(4) Find the free surface charge and free current density due to the potentials given in (2). (Assume that the dielectric constants and permeabilities on both sides of interface are $\varepsilon_o$ and $\mu_o$, respectively).
(a) Find the magnetic field at a distance $z$ above the center of a circular loop of radius $r$, which carries a steady current $I$, as shown below.

(b) A spherical shell of radius $R$, carrying a uniform surface charge $\sigma$, is set spinning at an angular velocity $\omega$, as shown below. Find the magnetic field at center $P$ of this spherical shell.
Charges \( \pm q \) and \(-q\) a distance \(d\) apart orbit around each other in the \(x-y\) plane \((z = 0)\), as shown below, at a frequency \( \omega \) \((d \ll c/\omega)\).

(a) The emitted radiation is primarily confined to one multipole. Which one?
(b) What is the angular distribution of the radiated power?
(c) What is the total power radiated?

The Lorentz transformation of the electromagnetic fields can be written

\[
E'_\parallel = E_\parallel, \quad B'_\parallel = B_\parallel
\]
\[
E'_\perp = \gamma(E_\perp + v \times B), \quad B'_\perp = \gamma(B_\perp - \frac{1}{c^2}v \times E_\parallel),
\]

Where \(\parallel\) indicates the component parallel to the velocity \(v\) and \(\perp\) the perpendicular part. For instance, \(E_\perp = E - E_\parallel\). As usual, \(\gamma = (1 - v^2/c^2)^{-1/2}\), where \(v = |v|\) is the speed.

(a) Show that the quantities \(E \cdot B\) and \(E^2 - c^2B^2\) are Lorentz invariants.
(b) Evaluate those quantities for a plane wave in vacuum.
radial direction: \( \frac{m \dot{v}^2}{R} = mg \cos \theta - N \)

Conservation of energy: \( mgR = mg R \cos \theta + \frac{1}{2} m \dot{v}^2 \) \((v_o = 0)\)

\[
N = 2gR \left(1 - \cos \theta \right)
\]

\[
N = mg \cos \theta - \frac{m \dot{v}^2}{R} = mg \cos \theta - \frac{m}{R} 2gR \left(1 - \cos \theta \right) = \\
= mg \left(3 \cos \theta - 2 \right) > 0
\]

(\(N \) being the normal force has to be positive. It vanishes precisely at the point where the particle "takes off".)

Thus, \(3 \cos \theta - 2 > 0\)

\[
\cos \theta_c = \frac{2}{3} 
\Rightarrow \theta_c = \cos^{-1} \left( \frac{2}{3} \right) \approx 48.2^\circ
\]
\[ T = \frac{1}{2} mr^2 \dot{\theta}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 \]

\[ V = -mg r \cos \theta + \frac{1}{2} kr^2 (r-b)^2 \]

\[ \frac{\partial L}{\partial \dot{\theta}} = -mgr \sin \theta \]

\[ \frac{\partial L}{\partial \dot{r}} = -mg r \cos \theta + k (r-b) \]

\[ \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\theta}, \quad \frac{\partial L}{\partial \ddot{\theta}} = m \ddot{\theta} \]

E.O.M.:

\[
\begin{align*}
\ddot{\phi} &= \phi_0^2 - m r \dot{\theta}^2 - mg \cos \theta + k (r-b) \\
\ddot{\theta} &= m r^2 \dot{\phi} + 2mr \dot{r} \dot{\theta} + mgs \sin \theta
\end{align*}
\]
1. When masses $m_1$ and $m_2$ are displaced from their equilibrium positions by an amount $x_1$ and $x_2$, respectively, the equations of motion are:

\[
\begin{align*}
M \ddot{x}_1 + (K + k_{12}) x_1 - k_{12} x_2 &= 0 \\
M \ddot{x}_2 + (K + k_{12}) x_2 - k_{12} x_1 &= 0
\end{align*}
\]  

(1)

2. We assume that the motion is oscillatory and attempt the following solution:

\[
\begin{align*}
x_1(t) &= B_1 e^{i\omega t} \\
x_2(t) &= B_2 e^{i\omega t}
\end{align*}
\]  

(2)

Combining equ. (1) and (2), we obtain:

\[
\begin{align*}
(K + k_{12} - M\omega^2) B_1 - k_{12} B_2 &= 0 \\
- k_{12} B_1 + (K + k_{12} - M\omega^2) B_2 &= 0
\end{align*}
\]  

(3)

For non-trivial solutions, we demand,

\[
\begin{vmatrix}
K + k_{12} - M\omega^2 & -k_{12} \\
- k_{12} & K + k_{12} - M\omega^2
\end{vmatrix} = 0
\]  

(4)
\[ w_1 = \sqrt{\frac{k}{M}} \]
\[ w_2 = \sqrt{\frac{K}{M}} \]

(3) For the symmetrical mode shown, the two masses are always moving "in phase", the spring "k1" is never involved in contributing to the potential term of the motion.

The associated frequency is: \[ w_1 = \sqrt{\frac{k}{M}} \]

For the anti-symmetrical mode, the two masses move "out-of-phase". And, the mode energy is higher.

\[ w_2 = \sqrt{\frac{K}{M}} \]
Suggested Solutions

Part I Mechanics
I-4 Gravity or other Central Potential

\[ m_1 \quad O \quad m_2 \]
\[ x_1 \quad O \quad x_2 \]
\[ r = |x_2 - x_1| \]

When two particles are initially at rest separated by a distance \( r_0 \), the system has the total energy

\[ E_0 = -G \frac{m_1 m_2}{r_0} \]  

(1)

The coordinates of the particles, \( x_1 \) and \( x_2 \), are measured from the position of the center of mass. At any time the total energy is

\[ E = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - G \frac{m_1 m_2}{r} \]  

(2)

and the linear momentum, at any time, is

\[ p = m_1 \dot{x}_1 + m_2 \dot{x}_2 = 0 \]  

(3)

From the conservation of energy we have \( E = E_0 \), or

\[ -G \frac{m_1 m_2}{r_0} = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - G \frac{m_1 m_2}{r} \]  

(4)

Using (3) in (4), we find

\[ \dot{x}_1 = v_1 = m_2 \sqrt{\frac{2G}{M} \left[ \frac{1}{r} - \frac{1}{r_0} \right]} \]

\[ \dot{x}_1 = v_2 = -m_1 \sqrt{\frac{2G}{M} \left[ \frac{1}{r} - \frac{1}{r_0} \right]} \]  

(5)
Write the energy and momentum conservation equations:

\[ p + p_{\text{ph}} = 0 \]

\[ mc^2 + \Delta E = \hbar \omega + \sqrt{p^2 c^2 + m^2 c^4} \]

where \( p \) is the momentum of the atom after emitting the photon, \( p_{\text{ph}} = \hbar \omega/c \), the momentum of the photon, and \( \omega \) is the photon frequency.

Substituting \( p = -\hbar \omega/c \) from the momentum conservation equation into the energy conservation equation above, and rewriting it into the form

\[ mc^2 + \Delta E - \hbar \omega = \sqrt{\hbar^2 \omega^2 + m^2 c^4} \]

we find after squaring both sides,

\[ \omega = \frac{\Delta E}{2\hbar} \left( \frac{\Delta E + 2mc^2}{\Delta E + mc^2} \right) \]

and taking into account that \( \Delta E + mc^2 = Mc^2 \) we may rewrite the above as

\[ \omega = \frac{\Delta E}{\hbar} \left( 1 - \frac{\Delta E}{2Mc^2} \right) \]

which is smaller by the amount of \( (\Delta E)^2/(2Mc^2 \hbar) \) than
it would have been without the relativistic effects. In the case of a crystalline lattice (Mössbauer effect), the atoms are strongly coupled to the lattice and have an effective mass \( M_0 \gg M \). From the equation above we can see that in this case the atom practically does not absorb energy, which all goes into the energy of the photon, and therefore there is no frequency shift due to this effect.
Electricity and Magnetism
I-6 Electrostatics or Boundary Value

(a) \( \sigma_a = \frac{-q_a}{4\pi a^2} \), \( \sigma_b = \frac{-q_b}{4\pi b^2} \), \( \sigma_R = \frac{q_a + q_b}{4\pi R^2} \).

(b) \( E_{\text{out}} = \frac{1}{4\pi \varepsilon_0} \frac{q_a + q_b}{r^2} \), where \( r \) = vector from center of large sphere.

(c) \( E_a = \frac{1}{4\pi \varepsilon_0} \frac{q_a}{r_a^2} \), \( E_b = \frac{1}{4\pi \varepsilon_0} \frac{q_b}{r_b^2} \), where \( r_a \) (\( r_b \)) is the vector from center of cavity \( a \) (\( b \)).

(d) Zero.
\( \mathbf{E} = -\mathbf{V} - \frac{\partial \mathbf{A}}{\partial t} \) \quad \ldots \quad (1)

(ii) \( \mathbf{B} = \nabla \times \mathbf{A} \) \quad \ldots \quad (2)

(2) from equ. (1), \( \mathbf{E} = -\mathbf{V} - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 \alpha}{2} (ct - |x|) \hat{z} \) \quad \ldots \quad (3)

from equ. (2), \( \mathbf{B} = \nabla \times \mathbf{A} = \pm \frac{\mu_0 \alpha}{2c} (ct - |x|) \hat{y} \) \quad \ldots \quad (4)

(plus for \( x < 0 \), minus for \( x > 0 \)).

(These are for \( |x| < ct \); when \( |x| > ct \), \( \mathbf{E} = 0 = \mathbf{B} \)).

(3) (i) The discontinuity in \( \mathbf{B} \) and \( \nabla \mathbf{f} \) is related:

\[ \nabla \mathbf{f} = \mathbf{B}_1 - \mathbf{B}_2 \] \quad \ldots \quad (5)

("\( \mathbf{f} \)" is the component that is perpendicular to the interface.

(ii) The discontinuity in \( \mathbf{B} \) and \( \mathbf{T}_n \) is related:

\[ \frac{1}{\mu_1} (B_1)_{\parallel} - \frac{1}{\mu_2} (B_2)_{\parallel} = \mathbf{T}_n \times \hat{n} \] \quad \ldots \quad (6)

("\( \parallel \)" is the component of \( \mathbf{B} \) that is parallel to the interface.
("\( \hat{n} \)" is the interface normal.)
1-1

(4) From equ. (4), \( \mathbf{B} \) has a discontinuity at \( x = 1 \).

From equ. (6), we find:

\[
\left( \frac{x_2}{2} - \frac{x_1}{2} \right) \hat{y} = \hat{k} \times \hat{x}
\]

\[\therefore \hat{k} = (x_1 - x) \hat{z}.\]

From equ. (3), \( \mathbf{E} \) is continuous.

There is no free surface charge.

\[\therefore \delta \mathbf{p} = 0\]

\[\#\]
By symmetry, the horizontal out-of-plane component of $dB$ produced by the segment $dl$ will be cancelled out as we integrate it over the entire current loop: $B_{II} = 0$

The vertical out-of-plane component of $dB$:

$$B = \frac{\mu_0 I}{4\pi} \int \frac{dl \cos \theta}{r^2}$$

$$B = \frac{\mu_0 I}{4\pi} \frac{2\pi r \cos \theta}{r^2} = \frac{\mu_0 I}{2} \frac{r^2}{(r^2 + z^2)^{3/2}}$$
Decompose the rotating spherical shell into a sequence of rotating rings, having a current component \( \text{d}I \):

(i) \( \text{d}I = (\vec{\sigma} \cdot \vec{V}) R \text{d}\theta \) \ldots \ldots  (3)

Here \( \vec{V} \) is the velocity, \( \vec{V} = \vec{V} \vec{T} \) \ldots \ldots  (4)

Combining (3) & (4), \[ \text{d}I = \sigma \omega R \text{d}\theta \ldots \ldots  (5) \]

(ii) From eqn.- (1), the \( \vec{B} \)-field at point- \( P \) is:

\[ B = \frac{\mu_0}{2} \int_0^\pi \int_0^{2\pi} (\sigma \omega + R \text{d}\theta) \frac{R^2}{R^3} \text{d}I \]

From eqn.- (5), \( B = \frac{\mu_0}{2} \int_0^\pi \int_0^{2\pi} (\sigma \omega + R \text{d}\theta) \frac{R^2}{R^3} \text{d}I \)

From eqn.- (2), \( B = \frac{\mu_0}{2} \sigma \omega R^3 \int_0^\pi \sin^3 \theta \text{d}\theta \)

\[ B = \frac{\mu_0}{2} \sigma \omega R \int_0^\pi \sin^3 \theta \text{d}\theta \]

\[ \therefore B = \frac{2}{5} \mu_0 \sigma \omega R \]
(a, b) At \( r \gg d \), the emitted radiation is confined to a dipole (\( r \gg \lambda \gg d \)) where \( \lambda \) is the wavelength. The vector potential of the system with dipole moment \( \vec{p} \) at a distance \( r \gg \lambda \) is given by
\[
\vec{A} = \frac{1}{c r_0} \vec{p}
\]
The magnetic field of the system is given by
\[
\vec{H} = \frac{1}{c} \left[ \vec{A} \times \hat{\mathbf{n}} \right] = \frac{1}{c^2 r_0} \vec{p} \times \hat{\mathbf{n}},
\]
where \( \vec{p} = q \vec{d} \) is the dipole moment of the system, \( \hat{\mathbf{n}} \) is the unit vector in the direction of observation, and \( r_0 \) is the distance from the origin as shown in the figure below:

The energy flux is given by the Poyting vector \( \vec{S} \):
\[
\vec{S} = c \frac{\vec{H}^2}{4\pi} \hat{\mathbf{n}}\]
\( [I-9] \) solutions - continued.

The radiated power in a solid angle \( d\Omega \) is given by

\[
\, dP = S r_o^2 \, d\theta = c H r_o^2 \, \frac{\hat{r} \cdot \hat{r}}{4\pi} \, d\theta
\]

and upon substitution we find

\[
\, dP = \frac{1}{4\pi c} \, |\vec{\hat{p}} \times \hat{n}|^2 \, d\theta
\]

Noting that \( \vec{p} = \hat{x} d = (p \cos \omega t, p \sin \omega t) = (p_x, p_y) \)
we have

\[
\vec{\hat{p}} \times \hat{n} = \left| \begin{array}{ccc}
\hat{x} & \hat{y} & \hat{z} \\
p_x & p_y & 0 \\
0 & 0 & \omega
\end{array} \right| \omega^2 = \left[ p_y n_z \hat{i} + p_x n_z \hat{j} - (p_x n_y - p_y n_x) \hat{k} \right] \omega^2
\]

\[
<|\vec{\hat{p}} \times \hat{n}|^2> = <p_x^2 \cos^2 \theta + p_y^2 \sin^2 \omega t \sin^2 \theta + p_z^2 \sin^2 \omega t \cos^2 \theta + p_x^2 \sin^2 \omega t \sin^2 \theta * \sin \phi \cos \phi> \omega^4
\]

Taking the average over the period of revolution:

\[
<\sin^2 \omega t> = <\cos^2 \omega t> = \frac{1}{2} \quad \text{and} \quad <2 \sin \omega t \cos \omega t> = <\sin 2\omega t> = 0 \quad \text{we find}
\]

\[
<|\vec{\hat{p}} \times \hat{n}|^2> = \frac{1}{2} p^2 \left( 1 + \cos^2 \theta \right) \omega^4
\]

\[
\therefore \langle dP \rangle = \frac{1}{4\pi c^2} \, \frac{1}{2} p^2 \omega^4 \left( 1 + \cos^2 \theta \right) \, d\Omega
\]
:: (b) \( \langle dP \rangle = \frac{p^2 \omega^4}{8\pi c^3} (1 + \cos^2 \theta) d\theta \)

(c) The total power radiated is

\[
\langle P \rangle = \int \frac{dP}{d\theta} d\theta = \frac{p^2 \omega^4}{8\pi c^3} 2\pi \int_0^\pi (1 + \cos^2 \theta) \sin \theta d\theta
\]

\[
\langle P \rangle = \frac{p^2 \omega^4}{4c^3} \left[ -\cos \theta - \frac{1}{3} \cos^3 \theta \right]_0^\pi = \frac{p^2 \omega^4}{4c^3} \left[ \pi + \frac{2}{3} \right]
\]

\[
\langle P \rangle = \frac{2p^2 \omega^4}{3c^3} = \frac{2q^2 \lambda^2 \omega^4}{3c^3}
\]
\[ E' \cdot B' = E_{\parallel} \cdot B_{\parallel} + \gamma^2 E_{\perp} \cdot B_{\perp} - \frac{\gamma^2}{c^2} (v \times B_{\perp}) \cdot (v \times E_{\perp}) + \gamma^2 v \times B_{\perp} \cdot E_{\perp} - \gamma^2 \frac{v^2}{c^2} B_{\perp} \cdot E_{\perp} - \frac{\gamma^2}{c^2} v \cdot E_{\perp} \times E_{\perp} \]

\[ = E_{\parallel} \cdot B_{\parallel} + \gamma^2 (1 - \frac{v^2}{c^2}) E_{\perp} \cdot B_{\perp} \]

\[ = E \cdot B \]

\[ E'^2 - c^2 B'^2 = E_{\parallel}^2 - c^2 B_{\parallel}^2 + \gamma^2 E_{\perp}^2 + \gamma^2 v^2 B_{\perp}^2 + \gamma^2 v^2 v \cdot B_{\perp} \times E_{\perp} - \gamma^2 \frac{v^2}{c^2} B_{\perp}^2 - \gamma^2 \frac{v^2}{c^2} E_{\perp}^2 + 2 \gamma^2 v \cdot \frac{E_{\perp} \times E_{\perp}}{c^2} \]

\[ = E^2 - c^2 B^2 \]

**Plane wave:**

\[ E = E_{y_0} e^{i (k \cdot r - \omega t)} \]

\[ B = B_{z_0} e^{i (k \cdot r - \omega t)} \]

\[ k \cdot E_{y_0} = 0 \]

\[ E \cdot B_{z_0} = 0 \]

\[ k \times E_{y_0} = \omega B_{z_0} \]

\[ k \times B = -\omega \frac{c^2}{E_{y_0}} \]
\[ \hat{E} \cdot \hat{B}^* = \hat{E}_{10} \cdot \hat{B}_{10} = \hat{E}_{10} \cdot \frac{\hat{E}_{10} \times \hat{E}_{10}^*}{\omega^2} = 0 \]

\[ E \cdot E^* - \frac{E^2}{\omega^2} = |\hat{E}_{10}|^2 - \frac{1}{\omega^2} (\hat{E} \times \hat{E}_{10}) \cdot (\hat{E} \cdot \hat{E}_{10}) \]

\[ = |\hat{E}_{10}|^2 - \frac{c^2 k^2}{\omega^2} |\hat{E}_{10}|^2 = 0 \]
Physics PhD Qualifying Examination
Part II – Friday, August 27, 2008

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Identification Number: ________________________________

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7. YOU MUST SHOW ALL YOUR WORK.
A particle is in a one-dimensional potential well given by \( V(x) = -e \delta(x) \), where \( \delta(x) \) is the Dirac delta function and \( e > 0 \) is a constant.
Find the energy and the normalized wave-function of the bound state(s).

Hint: You must carefully consider and study the possible discontinuity in the derivative of the wave function \( \psi'(x) \) at \( x = 0 \). You can do this by integrating Schrödinger’s equation for the above system from \( -\varepsilon \) to \( +\varepsilon \) and the let \( \varepsilon \to 0 \).
This is at the heart of this problem, and without a meaningful treatment and analysis of this discontinuity you will not pass this problem.

Consider a particle of mass \( m \) in a one-dimensional box with infinite high walls at \( x = 0 \) and \( x = L \).

(a) Find the eigenenergies \( E_n \) and normalized eigenfunctions \( \phi_n \) for the particle in this box.
(b) Calculate the first order correction to \( E_3^{(0)} \) for the particle due to the following perturbation:

\[
H' = 10^{-3} E_1 \frac{x}{L},
\]

where \( E_1 = \frac{\hbar^2 \pi^2}{2mL^2} \) is a constant.
(a) Recall that the raising and lowering operators for the angular momentum are given by

\[ L_\pm = L_\pm \pm iL_z. \]

Express \( L_\pm \) and \( L_\pm \) in terms of \( L_z \), and use this to prove that for any state with definite angular momentum quantum numbers \( \ell, m \), the following expectation values vanish:

\[ \langle L_\pm \rangle = \langle L_\pm \rangle = 0. \]

(b) Prove the following identities:

\[ 4L_z^2 = 2(L^2 - L_z^2) + L_z L_+ + L_z L_- \quad \text{and} \quad 4L_z^2 = 2(L^2 - L_z^2) - L_z L_+ - L_z L_- . \]

(c) Use the identities from (b) to compute the expectation values

\[ \langle L_\pm^2 \rangle, \quad \langle L_\pm^2 \rangle, \]

for any state with definite angular momentum quantum numbers \( \ell, m \).
(a) Evaluate the differential scattering cross-section in a repulsive potential, \( V(r) = A/r^2 \), in the Born approximation.

(b) Compare your above quantum results for the differential scattering cross section with the classical one, which is provided below for your convenience. Determine the limit of applicability for both cross sections (quantum and classical).

Classical Result [provided to you for part (b)]:
The differential scattering cross section for the classical mechanics case is given below; where we give the connection between the scattering angle \( \theta \) and the impact parameter \( \rho \):

\[
\int_{\rho} \frac{\mu v \rho d\rho}{r^2 \sqrt{2\mu(E-V) - (\mu v / r)^2}} = (\pi - \theta)/2.
\]

Here \( r_0 \) is the zero of the expression under the square root sign. Also \( v \) and \( \mu \) are the incident speed of the particle and the reduced mass, respectively.

For the classical treatment one can determine the differential scattering cross section by integrating the above equation. The results are:

\[
\rho^2 = \frac{A(\pi - \theta)^2}{E\theta(2\pi - \theta)},
\]

\[
d\sigma = -2\pi\rho \frac{d\rho}{d\theta} d\theta = \frac{2\pi^2}{E} \frac{\pi - \theta}{\theta^2(2\pi - \theta)^2} d\theta.
\]
(a) Write down the uncertainty relation of space and momentum.
(b) Use this uncertainty relation to estimate the ground-state energy of a one-dimensional simple harmonic oscillator with mass $m$ and angular frequency $\omega$. 

\[ H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 \]

A one dimensional simple harmonic oscillator of mass $m$ and angular frequency $\omega$ is acted upon by a spatially uniform but time-dependent force (NOT POTENTIAL)

\[ F(t) = \frac{F_0 \tau}{\omega (t^2 + \tau^2)} \]

At $t = -\infty$ the oscillator is known to be found in the ground state. Using time-dependent perturbation theory to first order, calculate the probability that the oscillator is found in the first excited state at $t = \infty$. 

\[ F(t) = \frac{F_0 \tau}{\omega (t^2 + \tau^2)} \]
The equation of state of a hypothetical ferromagnetic material is given by the implicit expression
\[ m = \tanh\left( \frac{Jm + B}{kT} \right), \]

where \( m = m(T, B) \) is the dimensionless magnetization (order parameter), \( B \) is the external magnetic field, \( T \) is the temperature, \( k \) is the Boltzmann constant, and \( J \) is a material-specific constant.

(a) What is the critical temperature \( T_c \) below which the system exhibits spontaneous magnetization? (We refer to spontaneous magnetization when \( m \neq 0 \) at \( B = 0 \).)

(b) Show that in the region just below \( T_c \), the spontaneous magnetization behaves as
\[ m(T, 0) \approx \text{const.} |T - T_c|^b, \]

and determine the value of the critical exponent \( b \).

Consider the Berthelot equation of state of a real gas:
\[ P = \frac{RT}{V - b} - \frac{a}{TV^2}, \]

where \( P \) is the pressure, \( T \) is the temperature, and \( V \) is the volume. \( R \) is the gas constant and \( a, b \) are empirical constants. The critical point of the Berthelot gas is characterized by critical pressure \( P_c \), critical volume \( V_c \) and critical temperature \( T_c \).

Find \( P_c, T_c \) and \( V_c \) in terms of \( R, a, \) and \( b \).
Consider a system of $N$ distinguishable non-interacting spins in a magnetic field $H$. Each spin has a magnetic moment of size $\mu$, and each can point either parallel or antiparallel to the field. Thus, the energy of a particular configuration is

$$E_{n_1, n_2, \ldots, n_N} = -\sum_{i=1}^{N} n_i \mu H, \quad n_i = \pm 1,$$

where $n_i \mu$ is the magnetic moment of spin $i$ in the direction of the field.

(a) Determine the average internal energy of this system as a function of $\beta (=1/kT)$, $H$ and $N$ by employing an ensemble characterized by these variables.

(b) Determine the entropy of this system as a function of $\beta$, $H$ and $N$.

(c) Determine the behavior of the average internal energy and entropy for this system as $T \to 0$.

Consider a hypothetical Fermi system with $N$ particles in volume $V$ and with the single-particle density of states $g(\varepsilon)$ given by

$$g(\varepsilon) = \begin{cases} 0 & \text{if } \varepsilon < 0 \\ \alpha V & \text{if } \varepsilon > 0 \end{cases},$$

where $\alpha$ is a constant.

Find the Fermi energy $\varepsilon_F$, the internal energy, and the pressure of the system at zero temperature.
\[ V(x) = -c \delta(x) \quad (c > 0) \]

(1) \[-\frac{\hbar^2}{2m} \psi''(x) + V(x) \psi(x) = E \psi(x) \]

\[ \psi''(x) = -\frac{2mE}{\hbar^2} \psi(x) \]

for bound state one needs \( \psi(x) \frac{\rightarrow}{x \to \pm \infty} 0 \),

thus \( E < 0 \)

(2)
\[ \psi(x) = \begin{cases} 
A e^{\kappa x} & x < 0 \\
B e^{-\kappa x} & x > 0 
\end{cases} \]

where \( \kappa = \sqrt{\frac{2m|E|}{\hbar^2}} \)

The heart of this problem is to determine the energy \( E \) of the bound state(s).

Integrating (1) from \( -\varepsilon \) to \( +\varepsilon \) and take \( \varepsilon \to 0 \):

\[ \psi''(x) = -\frac{2m}{\hbar^2} [E - V(x)] \psi(x) \]

\[ \int_{-\varepsilon}^{+\varepsilon} \psi''(x) \, dx = -\frac{2m}{\hbar^2} \left[ E \int_{-\varepsilon}^{+\varepsilon} \psi(x) \, dx - \int_{-\varepsilon}^{+\varepsilon} V(x) \psi(x) \, dx \right] \]

\( \psi(x) \) must be continuous everywhere (including at \( x = 0 \))

\[ \Rightarrow A = B \]
\[ \phi'(x) \bigg|_{-\infty}^{+\infty} = -\frac{2m}{\hbar^2} \left[ \Theta - \int_{-\infty}^{+\infty} \left(-c\varepsilon(x)\right) \phi'(x) \, dx \right] \]

\[ \phi'(+0) - \phi'(-0) = -\frac{2mc}{\hbar^2} \phi'(0) \]

From (2):

\[-\kappa A - \kappa A = -\frac{2mc}{\hbar^2} A\]

\[2\kappa A = \frac{2mc}{\hbar^2} A\]

\[\Rightarrow \kappa = \frac{mc}{\hbar^2}\]

and

\[|E| = \frac{\hbar^2 \kappa^2}{2m} = \frac{\hbar^2}{2m} \frac{m^2 c^2}{\hbar^4} = \frac{mc^2}{2\hbar^2}\]

\[\boxed{E = -\frac{mc^2}{2\hbar^2}}\]

a single bound state

From normalization:

\[\phi(x) = Ae^{-\kappa x^2} \]

\[\int_{-\infty}^{+\infty} |\phi(x)|^2 \, dx = 1 \Rightarrow \phi(x) = \sqrt{\kappa} e^{-\kappa x^2}\]
II.2 Perturbation Theory

(a) Eigenenergies \( E_n = n^2 E_1 \) with \( E_1 = \frac{\hbar^2 \pi^2}{2mL^2} \)

Eigenfunction \( \psi_n = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi x}{L} \right) \)

(b) First-order correction to \( E_3^{(0)} \)

\[
E_3^{(1)} = \langle \psi_3 | H' | \psi_3 \rangle = \frac{2}{L^2} 10^{-3} E_1 \int_0^L x \sin^2 \left( \frac{3\pi x}{L} \right) dx
\]

\[
\gamma = \frac{3\pi x}{L} = \frac{2}{L^2} 10^{-3} E_1 \left( \frac{L}{3\pi} \right)^2 \int_0^{3\pi} y \sin^2 y dy
\]

\[
x = \frac{L}{3\pi} y
\]

\[
dx = \frac{L}{3\pi} dy
\]

\[
\frac{2}{9\pi^2} 10^{-3} E_1 \left[ \frac{y^2}{4} - \frac{y \sin 2y}{4} - \frac{\cos 2y}{8} \right]_0^{3\pi} = 0
\]

\[
= \frac{2}{9\pi^2} 10^{-3} E_1 \left[ \frac{9\pi^2}{4} - \frac{3\pi \sin 6\pi}{4} - \frac{\sin 6\pi}{8} \right]
\]

\[
= \frac{2}{9\pi^2} 10^{-3} E_1 \left[ \frac{9\pi^2}{4} - \frac{1}{8} - \left( -\frac{1}{8} \right) \right]
\]

\[
E_3^{(1)} = 10^{-3} \frac{E_1}{2}
\]

* Integral 17.17.10 Schaum Outline Mathematical Handbook of Formulas and Tables, 2nd ed.*
Problem I-3

(a) \[ L_+ + L_- = 2L_x \]
\[ L_+ - L_- = 2iL_y \]
\[ \langle L_x \rangle = \frac{1}{2} \langle \ell m \mid L_+ + L_- \mid \ell m \rangle = 0 \]
\[ \langle L_y \rangle = \frac{1}{2} \langle \ell m \mid L_+ - L_- \mid \ell m \rangle = 0 \]
for the same reason.

(b) \[ L_+ L_- = L_x^2 - L_y^2 + i(L_xL_y + L_yL_x) \]
Thus \[ L_+L_+ + L_-L_- = 2(L_x^2 - L_y^2) \]
\[ L_x^2 + L_y^2 = L^2 - L_2 \]
\[ 4L_x^2 = 2(L_x^2 - L_y^2) + 2(L_x^2 + L_y^2) \]
\[ = 2(L^2 - L_2^2) + L_+L_+ + L_-L_- \]
\[ 4L_y^2 = 2(L_x^2 + L_y^2) - 2(L_x^2 - L_y^2) \]
\[ = 2(L^2 - L_2^2) - L_+L_+ - L_-L_- \]

(c) \[ \langle \ell m \mid L_+L_+ L_-L_- \mid \ell m \rangle \propto \langle \ell m \mid L, m \pm 2 \rangle = 0 \]
\[ \langle L_x \rangle = \langle L_y \rangle = \frac{1}{2} \langle L^2 - L_2^2 \rangle = \frac{\ell^2(\ell(\ell+1) - m^2)}{2} \]
\[ \text{(II-4) Solutions:} \]

The scattering amplitude is in the Born approximation given by the equation

\[ f_{\text{Born}}(\phi) = -\frac{\mu}{2\pi \hbar^2} \int e^{i(\vec{q} \cdot \vec{r})} V(r) d\tau = -\frac{\pi \mu}{\hbar^2} q \]

where \( \vec{q} = \vec{p}' - \vec{p} \), \( q = 2\hbar \sin \frac{1}{2} \phi \)

hence, \( d\sigma_{\text{Born}} = |f(\phi)|^2 d\Omega = \frac{\pi^3 \mu^2}{2\hbar^2 E} \cot \frac{1}{2} \phi \)

In classical mechanics, we have the following connection between the angle of scattering and the impact parameter \( p \):

\[ \frac{\mu \nu \rho}{r_0^2 \sqrt{[2\mu(E-V)-(\mu \nu \rho/r)^2]}} = \frac{\pi - \phi}{2} \]

where \( r_0 \) is the zero of the expression under the square root sign. Upon integration

\[ p^2 = \frac{\Delta}{E} \frac{1}{r_0^2} \frac{(\pi - \phi)^2}{2\pi - \phi} \]

\[ d\sigma = -2\pi \rho \frac{dp}{d\phi} d\Omega = \frac{2\pi^3 \Delta}{E} \frac{\pi - \phi}{\sqrt{2\pi - \phi}} d\phi \]

Now, if \( 8\mu \rho_0^2 \ll 1 \), we can apply
(1) The uncertainty relationship is:
\[ \Delta x \Delta p_x \geq \frac{\hbar}{2} \]

(2) The total energy of a harmonic oscillator:
\[ E = \frac{P^2}{2m} + \frac{1}{2} m \omega^2 x^2 \]

(iii) Using equ. 1, and taking the lowest limit:
\[ \Delta x \Delta p_x = \frac{\hbar}{2} \]

(iv) To estimate the ground state energy, we minimize equ. 1 and solve for (\Delta x).
\[ -\frac{\hbar}{2m(\Delta x)^2} + m \omega^2 \Delta x = 0 \]

\[ \Delta x = \sqrt{\frac{\hbar}{2m \omega}} \]

Substitute equ. 1 into equ. 2, we obtain:
\[ E = \frac{\hbar^2}{8m \left( \frac{\hbar}{2m \omega} \right)} + \frac{1}{2} m \omega^2 \left( \frac{\hbar}{2m \omega} \right) \]

\[ \Rightarrow E = \frac{\hbar \omega}{4} + \frac{1}{2} \hbar \omega = \frac{1}{2} \hbar \omega \]
Problem II-6

\[ F = -\frac{d}{dx} V = \text{const.} \Rightarrow V = -F_x \]

\[ x = \sqrt{\frac{\hbar}{2m \omega}} (a + a^+) \]

\[ \langle 1 | V | 10 \rangle = -\sqrt{\frac{\hbar}{2m \omega}} F(t) \langle 1 | a + a^+ | 10 \rangle = -\sqrt{\frac{\hbar}{2m \omega}} F(t) \]

\[ c^{(1)}(\infty) = \frac{t}{\hbar} \sqrt{\frac{\hbar}{2m \omega}} \int_{-\infty}^{\infty} dt \ F(t) \ e^{i \omega t} \]

\[ \int_{-\infty}^{\infty} dt \ F(t) = \frac{F_0 \pi}{\omega} e^{\omega t} \text{ by a variety of methods, including tables.} \]

\[ c^{(1)}(\infty) = \frac{t}{\hbar} F_0 \pi e^{-\omega t} \]

\[ \sqrt{2m \omega^3 \hbar} \]

\[ \text{Prob.} (0 \to 1) = \frac{(F_0 \pi)^2 e^{-2\omega t}}{2m \omega^3 \hbar} \]
II-2

\[ m = \tanh \left( \frac{Jm + B}{kT} \right) \]

Spontaneous magnetization: \( m \neq 0 \) when \( B = 0 \)

a) \[ m = \tanh \left( \frac{3}{kT} m \right) \]

\( \tanh(x) = x - \frac{1}{3} x^3 + \]

"graphical solution:"

\[ \frac{J}{kT} < 1 \]

No spontaneous magnetization

\[ \frac{J}{kT} > 1 \]

b) \( \frac{J}{kT} > 1 \) (\( T < T_c \))

There are 2 non-trivial solutions corresponding to spontaneous magnetization.

Thus, the critical temperature: \( \frac{J}{kT_c} = 1 \)

\[ T_c = \frac{J}{k} \]

For \( T \leq T_c : \quad (B = 0) \)

\[ m \sim \frac{3}{kT} m - \frac{1}{3} \left( \frac{2}{kT} \right)^3 m^3 \quad (m \neq 0) \]

\[ 1 = \frac{T_c}{T} - \frac{1}{3} \left( \frac{T_c}{T} \right)^3 m^2 \]

\[ m = \sqrt{3 \frac{T_c}{T} \left( \frac{T_c}{T} - 1 \right)} = \sqrt{3 \frac{T_c - T}{T} \frac{T_c - T}{T} \sim \sqrt{3 \frac{T_c - T}{T} - \frac{T_c - T}{T} \frac{T_c - T}{T}} \sim \sqrt{3 \frac{T_c - T}{T}} - \sqrt{3 \frac{T_c - T}{T} \frac{T_c - T}{T}} \sim \sqrt{3 \frac{T_c - T}{T} - \frac{T_c - T}{T} \frac{T_c - T}{T}} \sim \sqrt{3 \frac{T_c - T}{T}} \]

\[ m = \sqrt{3 \frac{T_c - T}{T_c}} \sim \sqrt{3 (T_c - T)^{1/2}} \quad (B = 0) \]

\[ \beta = \frac{1}{2} \]

\[ T_c \rightarrow T \]
\[ P_c = \frac{3}{2} \frac{T_c}{(V_c - b)^{-1} - \frac{1}{2} \frac{T_c}{V_c^{-1}} V_c^{-2}} \quad (1) \]

\[ \frac{\partial P}{\partial V_c} = 0 = -RT_c(V_c - b)^{-2} + 2aT_c^{-1}V_c^{-3} \quad (2) \]

\[ \frac{\partial^2 P}{\partial V_c^2} = 0 = 2RT_c(V_c - b)^{-3} - 6aT_c^{-1}V_c^{-4} \quad (3) \]

(2) divided by (3)

\[ 2(V_c - b)^{-1} = 3V_c^{-1} \]

\[ \Rightarrow V_c = 3b \]

From (1)

\[ T_c^2 = \frac{2aV_c^{-3}}{R(V_c - b)^{-2}} \Rightarrow T_c = \frac{8a}{V_c^{2} + 6bR} \]

\[ T_c, V_c \text{ in (1)} \Rightarrow P_c = \frac{1}{12b} \sqrt{\frac{2aR}{3b}} \]
For $N$ spins in a field $H$,

$$E_{m_1, m_2, \ldots, m_N} = \sum_{i=1}^{N} (-\mu_i \cdot H), \quad m_i = \pm 1$$

(a) $\beta$, $H$, and $N$ are constant in our ensemble.

$$\mathcal{Q} = \sum_{m_1, m_2, \ldots, m_N} e^{-\beta E_{m_1, m_2, \ldots, m_N}}$$

$$\mathcal{Q} = \sum_{m_1, m_2, \ldots, m_N} e^{-\beta \sum_{i=1}^{N} (-\mu_i \cdot H)}$$

$$\mathcal{Q} = \sum_{m_1, m_2, \ldots, m_N} \prod_{i=1}^{N} e^{\beta \mu_i \cdot H} = \prod_{i=1}^{N} \sum_{n=\pm 1} e^{\beta \mu_i \cdot H n}$$

since the factor for each spin is identical, so

$$\mathcal{Q} = (e^{\beta \mu H} + e^{-\beta \mu H})^N = (2 \cosh (\beta \mu H))^N.$$

$$\langle E \rangle = \frac{\partial \ln \mathcal{Q}}{\partial (-\beta)} = N \mu H \frac{e^{\beta \mu H} + e^{-\beta \mu H}}{e^{\beta \mu H} + e^{-\beta \mu H}}$$

$$\langle E \rangle = N \mu H \tanh (\beta \mu H) = -N \mu H \tanh (\beta \mu H)$$

(b) $\ln \mathcal{Q} = \frac{S}{\hbar_B} - \frac{E}{\hbar_B T} = -\beta H$
(2)

[II-9] solution-continued.

\[ S' = -k_B \ln \theta + k_B \beta \langle E \rangle \]

\[ S' = N k_B \left[ \ln (e^{\beta \mu H} + e^{-\beta \mu H}) - \beta \mu H \tanh (\beta \mu H) \right] \]

(c) In the limit \( T \to 0 \), or \( \beta \to \infty \), \( \tanh (\beta \mu H) \to 1 \), hence, \( \langle E \rangle \to -N\mu H \), i.e. the ground state has all spins \( + \), or aligned with the field.

\[ \lim_{\beta \to \infty} N k_B \left[ \ln (e^{\beta \mu H} + e^{-\beta \mu H}) - \beta \mu H \tanh (\beta \mu H) \right] \]

\[ = N k_B \left[ \beta \mu H - \beta \mu H \right] = 0 \], hence,

\[ S' \to 0 \]
Fermi system

\[ f(\varepsilon) = \begin{cases} \varepsilon & \text{if } \varepsilon < 0 \\ \frac{1}{\alpha N} & \text{if } \varepsilon > 0 \end{cases} \]

\[ T = 0 \quad \Rightarrow \quad n(\varepsilon) = \begin{cases} 1 & \varepsilon > \varepsilon_F \\ 0 & \varepsilon < \varepsilon_F \end{cases} \]

\[ N = \int_0^\infty n(\varepsilon) g(\varepsilon) \, d\varepsilon = \int_0^{\varepsilon_F} g(\varepsilon) \, d\varepsilon = \alpha V \varepsilon_F \]

\[ \varepsilon_F = \frac{1}{\alpha} \frac{N}{V} \]

\[ E_0 = \int_0^\varepsilon_F n(\varepsilon) g(\varepsilon) \, d\varepsilon = \int_0^{\varepsilon_F} \varepsilon g(\varepsilon) \, d\varepsilon = \alpha V \varepsilon_F^2 = \frac{\alpha V}{2} \left( \frac{1}{\alpha} \frac{N}{V} \right)^2 \]

\[ E_0 = \frac{N}{2\alpha} \frac{N}{V} \quad \frac{E_0}{N} = \frac{1}{2\alpha} \frac{N}{V} \]

\[ F(T, V, N) = E - TS \quad P = -\left( \frac{\partial F}{\partial V} \right)_{T, 0, N} = -\left( \frac{\partial E_0}{\partial V} \right)_N \]

\[ \quad = \frac{1}{2\alpha} \left( \frac{N}{V} \right)^2 \]