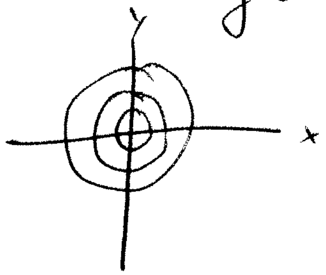
Exam #1 : Tuesday, Oct. 6, 2009

1: A velocity field is described by the equation $\vec{v} = y\hat{i} - x\hat{j}$.

a) Obtain an expression for the speed (magnitude of $\vec{v}$) as a function of x and y.

$$|\vec{v}| = (\vec{v}\cdot\vec{v})^{1/2} = (x^2+y^2)^{1/2} = r$$

b) Carefully draw curves of constant speed on the graph.



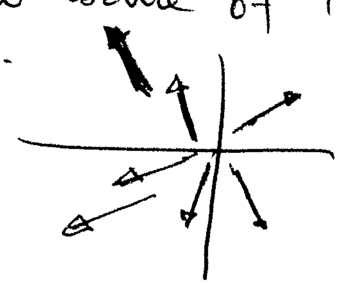
c) Calculate the gradient of this speed.

$$S \equiv |\vec{v}|$$

$$\vec{\nabla} S \equiv \vec{\nabla}|\vec{v}| = \frac{\partial}{\partial x}(x^2+y^2)^{1/2}\,\hat{i} + \frac{\partial}{\partial y}(x^2+y^2)^{1/2}\,\hat{j} = \frac{x}{(x^2+y^2)^{1/2}}\hat{i} + \frac{y}{(x^2+y^2)^{1/2}}\hat{j}$$

$$= \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j}$$

d) Draw some of these gradient vectors on the provided graph.

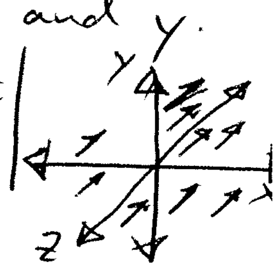


All point radially outward with equal magnitude, since $(\vec{\nabla}S\cdot\vec{\nabla}S)^{1/2} = \left(\frac{x^2}{r^2} + \frac{y^2}{r^2}\right)^{1/2} = \left(\frac{x^2+y^2}{r^2}\right)^{1/2} = \left(\frac{r^2}{r^2}\right)^{1/2} = 1$

e) obtain the curl of the field $\vec{v}$ as a function of x and y.

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & 0 \end{vmatrix} = \hat{i}\cancel{\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & 0 \end{vmatrix}}^{\,0} + \hat{j}\cancel{\begin{vmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\ 0 & y \end{vmatrix}}^{\,0} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y & -x \end{vmatrix}$$

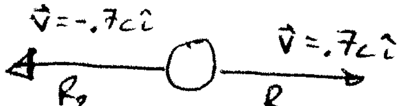
$$= \hat{k}\left(-\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}y\right) = -2\hat{k}$$



f) Draw the curl at several points in the x-y on the provided graph.

2. Two rockets, R_1 and R_2 leave Earth at $t=0s$, traveling in opposite directions with $.7c$ relative to Earth.
An observer on the Earth sees two events at times $t_1 = 1\mu s$ and $t_2 = 2\mu s$ and positions $x_1 = 1km$ and $x_2 = 10km$.

a) What is the time measured on R_1 for each event?



$$t'_a = \gamma\left(t_a - \frac{v}{c^2}x_a\right) \qquad \gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-.7^2}} = 1.4$$

$$t_1' = \gamma\left(t_1 - \frac{v}{c^2}x_1\right) = 1.40\left(10^{-6}s - \frac{.7}{c}\cdot 10^3 m\right)$$
$$= 1.40\left(10^{-6}s - 2.33\times10^{-6}s\right) = -1.86\times10^{-6}s$$
$$= -1.86\mu s$$

$$t_2' = \gamma\left(t_2 - \frac{v}{c^2}x_2\right) = 1.40\left(2\times10^{-6}s - \frac{.7}{c}\cdot 10\times10^3 m\right)$$
$$= 1.40\left(2\times10^{-6}s - 23.3\times10^{-6}s\right) = -29.8\times10^{-6}s$$
$$= -29.8\mu s$$

b) What is the time measured on R_2 for each event?

$$\gamma = 1.40 \qquad t_1'' = \gamma\left(t_1 - \frac{v}{c^2}x_1\right) = 1.40\left(10^{-6}s + \frac{.7}{c}10^3 m\right)$$
$$= 1.4\left(10^{-6}s + 2.33\times10^{-6}s\right) = 4.66\times10^{-6}s$$
$$= 4.66\mu S$$

$$t_2'' = \gamma\left(t_2 - \frac{v}{c^2}x_2\right) = 1.40\left(2\times10^{-6}s + \frac{.7}{c}10\times10^3 m\right)$$
$$= 1.40\left(2\times10^{-6}s + 2.33\times10^{-5}s\right) = 35.4\times10^{-6}s$$
$$= 35.4\mu S$$

c) What is the spacetime invariant ΔS^2?

$$\Delta S^2 = -c^2 \Delta t^2 + \Delta x^2 = -c^2 (t_2 - t_1)^2 + (x_2 - x_1)^2$$

$$= -9 \times 10^{16} (2\mu s - 1\mu s)^2 \frac{m^2}{s^2} + (10km - 1km)^2$$

$$= -9 \times 10^{16} \times 10^{-12} s^2 \frac{m^2}{s^2} + (9km)^2$$

$$= -9 \times 10^{+4} m^2 + 81 \times 10^6 m^2$$

$$= (81 \times 10^6 - 9 \times 10^4) m^2$$

$$= (81 \times 10^6 - 0.09 \times 10^6) m^2$$

$$= (80.91 \times 10^6) m^2$$

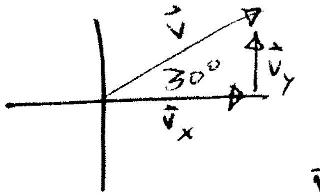
d) What does this say about causality?

Since $\Delta S^2 > 0$ it is a spacelike separation.

These events cannot be causaly related.

3) A particle is moving in the x-y plane at a speed .8c at an angle +30° relative to the x-axis.

a) Write the velocity in terms of unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$.



$\vec{V} = V_x \hat{i} + V_y \hat{j}$, $V_x = |\vec{V}| \cos\theta$, $V_y = |\vec{V}| \sin\theta$

$|\vec{V}| = .8c$

$\vec{V} = |V| \cos\theta \, \hat{i} + |V| \sin\theta \, \hat{j}$

$= .8c \cdot \frac{\sqrt{3}}{2} \hat{i} + .8c \frac{1}{2} \hat{j} = .4c\sqrt{3}\, \hat{i} + .4c \hat{j}$

As viewed from an observer moving with a speed .4c along x-axis

b) Obtain the components of speed along the x and y axes, as calculated by the non-relativistic transformation.

$u_x' = u_x - v$ $\qquad u_y' = u_y = .4c$

$= .4c\sqrt{3} - .4c$

$= .293c$

c) Obtain the components of speed along the x and y axes, as calculated by the relativistic Lorentz transformation.

$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{.4c\sqrt{3} - .4c}{1 - \frac{.4c\sqrt{3} \cdot .4c}{c^2}} = \frac{.293c}{1 - .4 \cdot .4\sqrt{3}} = .405c$$

$$u_y' = \frac{u_y}{\gamma(1 - \frac{u_x v}{c^2})} = \sqrt{1 - \frac{v^2}{c^2}} \, \frac{.4c}{.723} = \sqrt{1 - .16} \cdot .553c = .507c$$

d) What is the angle of motion in the moving frame as calculated in the non-relativistic case? (relative to x-axis)

$\tan\theta = (u_y/u_x)$ $\qquad \theta = \arctan\left(\frac{.4c}{.293c}\right) = \arctan(1.365) = 53.77°$

e) In the Lorentz transformation case, what is θ?

$\theta = \arctan\left(\frac{.507c}{.405c}\right) = \arctan(1.252) = 51.38°$

4. A neutrally charged rho meson ρ^0, which has mass $mc^2 = 770\,MeV$ decays at rest into two pi mesons, one positively charged π^+ and negatively charged π^- each having $mc^2 = 140\,MeV$.

a) What is the total energy of the system?

$$\mathcal{E}_T = \sqrt{p^2c^2 + m^2c^4} = \sqrt{0 + m^2c^4} = mc^2 = 770\,MeV$$

b) Calculate the energy of each pion.

$$\bar{\mathcal{E}}_\rho = \bar{\mathcal{E}}_{\pi^+} + \bar{\mathcal{E}}_{\pi^-} \Rightarrow \begin{pmatrix} 770\,MeV \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{p^2c^2 + m^2c^4} \\ p \end{pmatrix} + \begin{pmatrix} \sqrt{p^2c^2 + m^2c^4} \\ -p \end{pmatrix} = \begin{pmatrix} 2\mathcal{E}_\pi \\ 0 \end{pmatrix}$$

$$\mathcal{E}_\pi = \frac{770\,MeV}{2} = 385\,MeV$$

c) Calculate the momentum of each pion.

$$(385\,MeV)^2 = p^2c^2 + m^2c^4$$

$$pc = \sqrt{(385\,MeV)^2 - (140\,MeV)^2} = (148225 - 19600)^{1/2}\,MeV$$
$$= 358.6\,MeV$$

$$p = 358.6\,\frac{MeV}{c}$$

d) Calculate the velocity of each pion.

$$\mathcal{E} = \gamma mc^2 = \frac{\gamma mvc^2}{v} = \frac{pc^2}{v} \Rightarrow v = \frac{pc}{\mathcal{E}}c = \frac{358.6\,MeV}{385\,MeV}c = 0.93c$$