1. In the photoelectric effect. A photon of wavelength $\lambda$ imparts all of its energy to a free electron in a metal. Assume an incident photon has a wavelength of 440 nm. (25 pts.)

a. What color is it? (1 pt.)

**Indigo/Blue**

b. What is its energy? (8 pts.)

$$E = h\nu = \frac{hc}{\lambda} = \frac{1240eV\text{nm}}{440\text{nm}} = 2.82eV$$

c. Find the wavelength of an electron after absorbing the photon. (8 pts.)

The kinetic energy of an electron after absorbing a photon of such an energy is:

$$K = h\nu = 2.82eV$$

$$K = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2}$$

$$\lambda^2 = \frac{h^2}{2mK} = \frac{1240eV\text{nm}}{1240eV\text{nm}} = 0.730\text{nm}$$

d. Suppose the separation potential between the electron and the metal’s surface is 1 V. What will be the wavelength of the electron upon emergence from the metal? (8 pts.)

$$\lambda = h = \frac{hc}{\sqrt{2mc^2K}} = \frac{1240eV\text{nm}}{\sqrt{2} \times 0.511\text{MeV} \times 2.82eV} = 0.909\text{nm}$$
2. An atom of mass $m$ is attached to another by a one-dimensional harmonic oscillator potential energy with spring constant $K$, defined as $F=-kx$. (24 pts. + 1 free pt.)

a. Write the potential energy. (6 pts.)

$$\nabla U = F = -kx$$

$$U = -\int kx \, dx = \frac{1}{2} kx$$

b. Write the one dimensional time independent Schrödinger equation for $\psi(x)$ for a particle with the harmonic oscillator potential. (6 pts.)

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} Kx^2 \psi = E \psi$$

or $$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} m\omega^2 x^2 \psi = E \psi$$

c. In the space provided, carefully draw a diagram of the wave function $\psi(x)$ and probability $P(x)$ for the two lowest energy states. (6 pts.)

![Diagram showing two lowest energy states of a harmonic oscillator](image)

d. The wave function for the ground state is $\psi_0(x) = C_0 e^{-a^2 x^2/2}$. By direct substitution, find $a$ and the energy corresponding to this state. (6 pts.)

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} Kx^2 \psi = E \psi$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 C_0}{\partial x^2} e^{-a^2 x^2/2} + \frac{1}{2} Kx^2 C_0 e^{-a^2 x^2/2} = EC_0 e^{-a^2 x^2/2}$$

$$\frac{\hbar^2}{2m} C_0 e^{-a^2 x^2/2} \frac{\partial^2}{\partial x^2} a^2 x \frac{\partial C_0}{\partial x} e^{-a^2 x^2/2} + \frac{1}{2} Kx^2 C_0 e^{-a^2 x^2/2} = EC_0 e^{-a^2 x^2/2}$$

$$\frac{\hbar^2}{2m} (C_0 e^{-a^2 x^2/2} \alpha^2 - C_0 e^{-a^2 x^2/2} \alpha^4 x^2) + \frac{1}{2} Kx^2 C_0 e^{-a^2 x^2/2} = EC_0 e^{-a^2 x^2/2}$$
\[
\frac{\hbar^2}{2m} (\alpha^2 - \alpha^4 x^2) + \frac{1}{2} Kx^2 = E \\
\frac{\hbar^2}{2m} \alpha^4 x^2 = \frac{1}{2} Kx^2 = \frac{1}{2} m\omega^2 x^2 \\
\alpha^4 = \frac{m^2 \omega^2}{\hbar^2} \\
\alpha = \sqrt[4]{\frac{m\omega}{\hbar}} \\
E = \frac{\hbar^2}{2m} \alpha^2 = \frac{\hbar^2}{2m} \frac{m\omega}{\hbar} = \frac{\hbar \omega}{2}
\]
3. An electron is confined to move freely in a two-dimensional circular box of radius \(a = 1.0\) nm having infinite potential walls. The gradient vector in polar coordinates is \(\vec{V} = \hat{r} \frac{\partial}{\partial r} + \frac{1}{r} \hat{\phi} \frac{\partial}{\partial \phi}\). (25 pts.)

a. Obtain the Laplacian \(\nabla^2 = \vec{\nabla} \cdot \vec{\nabla}\) (0 pts.)

We must be careful because the unit vectors \(\hat{r}\) and \(\hat{\phi}\) depend on the coordinate \(\phi\).

Viz. \(\hat{r} = (\cos \phi \hat{x} + \sin \phi \hat{y})\) and \(\hat{\phi} = (- \sin \phi \hat{x} + \cos \phi \hat{y})\)

This results in \(\hat{\phi} \cdot \hat{\phi} = 1, \hat{r} \cdot \hat{r} = 1, \) and \(\hat{r} \cdot \hat{\phi} = 0\)

However \(\frac{\partial \hat{\phi}}{\partial \phi} = - \hat{r}, \) and \(\frac{\partial \hat{r}}{\partial \phi} = \hat{\phi}\)

Proceeding VERY CAREFULLY:

\[
\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \left(\hat{r} \frac{\partial}{\partial r} + \frac{1}{r} \hat{\phi} \frac{\partial}{\partial \phi}\right) \cdot \left(\hat{r} \frac{\partial}{\partial r} + \frac{1}{r} \hat{\phi} \frac{\partial}{\partial \phi}\right) = \\
\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \phi^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{r} \frac{\partial \hat{\phi}}{\partial \phi} \frac{\partial}{\partial r} + \frac{\partial \hat{r}}{\partial \phi} \frac{\partial}{\partial \phi} - \frac{\partial \hat{\phi}}{\partial \phi} \frac{\partial \hat{r}}{\partial \phi}
\]

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \phi^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{r} \frac{\partial}{\partial r}
\]

b. Write the time independent Schrödinger equation in 2-dimensional polar coordinates. (5 pts.)

\[
\frac{-\hbar^2}{2m} \nabla^2 \psi(r, \phi, t) + V(r) \psi(r, \phi, t) = E \psi(r, \phi, t)
\]

\[
\frac{-\hbar^2}{2m} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{r} \frac{\partial}{\partial r}\right) \psi + V(r) \psi = E \psi
\]

\[
V(r) = \begin{cases} 
0 & \text{for } r < a \\
\infty & \text{for } r > a
\end{cases}
\]

c. Using separation of variables technique obtain the \(r\) and \(\phi\) dependent wave equations for \(R(r)\) and \(\Phi(\phi)\), separately. (5 pts.)

Assume that the solution can be written in the form: \(\psi(r, \phi) = R(r) \Phi(\phi)\)

\[
\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{r} \frac{\partial}{\partial r}\right) R(r) \Phi(\phi) = -\frac{2mE}{\hbar^2} R(r) \Phi(\phi)
\]

\[
\frac{\partial^2}{\partial r^2} R(r) \Phi(\phi) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} R(r) \Phi(\phi) + \frac{1}{r} \frac{\partial}{\partial r} R(r) \Phi(\phi) = -\frac{2mE}{\hbar^2} R(r) \Phi(\phi)
\]

\[
\frac{r^2}{R(r)} \frac{\partial^2}{\partial r^2} R(r) + \frac{1}{\Phi(\phi) \Phi(\phi) \Phi(\phi)} \frac{\partial^2}{\partial \phi^2} \Phi(\phi) + \frac{1}{R(r)} \frac{1}{r} \frac{\partial}{\partial r} R(r) = -\frac{2mE}{\hbar^2} r^2
\]
\[
\frac{r^2}{R(r)} \frac{\partial^2}{\partial r^2} R(r) + \frac{2mE}{\hbar^2} r^2 + \frac{r}{R(r)} \frac{\partial}{\partial r} R(r) = -\frac{1}{\Phi(\varphi)} \frac{\partial^2}{\partial \varphi^2} \Phi(\varphi) = m^2
\]

So then we have:

\[
r^2 \frac{\partial^2 R(r)}{\partial r^2} + \frac{\partial R(r)}{\partial r} + \frac{2mE}{\hbar^2} r^2 R(r) = m^2 R(r)
\]

And:

\[
\frac{\partial^2}{\partial \varphi^2} \Phi(\varphi) = -m^2 \Phi(\varphi)
\]

d. What are the boundary conditions on \(R(r)\) and \(\Phi(\varphi)\)? (5 pts.)

The boundary conditions are:

\[R(a) = 0 \text{ and } \Phi(\varphi) = \Phi(\varphi + n2\pi)\]

Where \(n \in \mathbb{Z}\)

e. Obtain a solution for \(\Phi(\varphi)\). (5 pts.)

Recognize that this is a second order differential equation which yields the general solution:

\[\Phi(\varphi) = A e^{im\varphi} + B e^{-im\varphi}\]

f. Determine the quantum conditions by applying boundary conditions on \(\Phi(\varphi)\). (5 pts.)

We will use the boundary condition \(\Phi(\varphi) = \Phi(\varphi + n2\pi)\):

\[\Phi(\varphi) = A e^{im\varphi} + B e^{-im\varphi} = A e^{im(\varphi + n2\pi)} + B e^{-im(\varphi + n2\pi)} = A e^{im\varphi + in\pi} + B e^{-im\varphi - in\pi}\]

\(e^{imn2\pi}\) and \(e^{-imn2\pi}\) must both be equal to one. For this to be satisfied, the argument of the exponential must be equal to \(i\) times an integer multiple of \(2\pi\).

The only way to satisfy this criterion to require \(m \in \mathbb{Z}\). That is the quantum condition.
4. Assume $^{236}U$ undergoes symmetric fission into two uniformly charged spheres, each of radius R=5 fm. (10 pts.)

   a. Calculate the Coulomb energy in MeV when they are just touching. (5 pts.)

   \[ U = \frac{Q^2}{4\pi\varepsilon_0 R} = \frac{(46e)^2}{4\pi\varepsilon_0 \times 10 \text{ fm}} = \frac{(46 \times 1.6 \times 10^{-19} \text{ C})^2}{4\pi \times 8.85 \times 10^{-12} \text{ F/m} \times 10 \times 10^{-15} \text{ m}} = 304.5 \text{ MeV} \]

   b. Explain why $^{235}U + n$ fissions whereas $^{238}U + n$ does not. (5 pts.)

   $^{235}U$ has an even-odd pairing of protons and neutrons, whereas $^{238}U$ has an even-even pairing of protons and neutrons. If the former accepted a neutron, it would bring itself to an even-even nucleus, increase the stability and, by difference in binding energy, release energy. If the latter were to accept a neutron, it would bring itself to an even-odd nucleus, which is less energetically favorable, releasing not enough energy.
5. The distribution of electron energy levels in a star is given by \( \frac{dN_e}{dE} = \frac{8 \pi V}{c^3 h^3} E^2 \), where \( V \) is the volume of the star. (15 pts.)

a. For a star at temperature 0 K obtain an expression for the average energy \( \langle E \rangle \) of an electron assuming the Fermi energy is \( E_F \). (5 pts.)

\[
\langle E \rangle \equiv \frac{\int_0^\infty \frac{dN_e}{dE} E \, dE}{\int_0^\infty \frac{dN_e}{dE} \, dE} = \frac{\int_0^{E_F} \frac{dN_e}{dE} E \, dE}{\int_0^{E_F} \frac{dN_e}{dE} \, dE} = \frac{\int_0^{E_F} \frac{8 \pi V}{c^3 h^3} E^3 \, dE}{\int_0^{E_F} \frac{8 \pi V}{c^3 h^3} E^2 \, dE} = \frac{3 E_F^4}{4 E_F^3} = \frac{3}{4} E_F
\]

b. Assuming there are \( N_e \) electrons, find the Fermi energy by integrating \( \frac{dN_e}{dE} \). (5 pts.)

\[
N_e = \int_0^\infty \frac{8 \pi V}{c^3 h^3} E^2 \, dE = \int_0^{E_F} \frac{8 \pi V}{c^3 h^3} E^2 \, dE = \frac{8 \pi V}{3 c^3 h^3} E_F^3
\]

\[
E_F = \sqrt[3]{\frac{3 N_e c^3 h^3}{8 \pi V}} = c h \sqrt[3]{\frac{3 N_e}{8 \pi V}}
\]

c. If the number of nucleons in the Sun is \( N_n = 1.2 \times 10^{57} \), and the Sun becomes a white dwarf of radius \( R = 7 \times 10^3 km \), what will be the Fermi energy and average energy of the electrons? (5 pts.)

\[
V = \frac{4}{3} \pi R^3 = \frac{4}{3} \pi (7 \times 10^6 m)^3 = 1.437 \times 10^{21} m^3
\]

\[
N_e = \frac{N_n}{2} = 6.0 \times 10^{56}
\]

\[
E_F = c h \sqrt[3]{\frac{3 N_e}{8 \pi V}} = 1240 eV nm \times \sqrt[3]{\frac{3 \times 6.0 \times 10^{56}}{8 \pi \times 1.437 \times 10^{48} m^3}} = 4.56 \times 10^6 eV
\]

\[
\langle E \rangle = \frac{3}{4} E_F = 3.42 \times 10^5 eV
\]