ASTR-4240 — Gravitation & Cosmology
PHYS-4240 — General Relativity

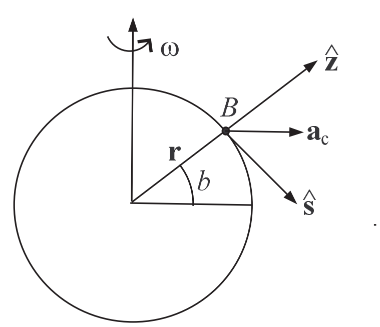
Class 2
Equivalence Principle

Exercise (30 pts)

1. (10 pts) — In spherical coordinates centered on the Earth, what are the radial and tangential components of the centrifugal acceleration at Budapest? The latitude of Budapest is 47.433°.

2. (10 pts) — By how much does the plumb line at Budapest deviate from the radial direction? Assume that the inertial and gravitational masses are equal. Give your answer in degrees.

3. (10 pts) — Suppose that plumb bob no. 1 has $m_{\rm I} = m_{\rm G}$ and plumb bob no. 2 has $m_{\rm I} - m_{\rm G} = 10^{-9} m_{\rm I}$. What is the angular difference between the corresponding plumb lines?



Solution

1. — Adopt a rotating coordinate system fixed to the Earth with its orgin at the Earth's center. The figure above is a cross section showing Budapest at position **r**. The vertical direction is $\hat{\boldsymbol{z}}$ and south is $\hat{\boldsymbol{s}}$. The centrifugal acceleration, $\mathbf{a}_\mathrm{c} = -\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$, lies along the direction indicated. Its magnitude is

$$a_\mathrm{c} = \omega^2\, r\, \sin\left(90^\circ - b\right), \tag{1}$$

where $r = 6.38 \times 10^8\,\mathrm{cm}$ is the radius of the Earth, $\omega = 7.27 \times 10^{-5}\,\mathrm{s}^{-1}$ is its angular velocity, and $b = 47.433^\circ$ is the latitude of Budapest. Plugging in the numbers, we find $a_c = 2.28\,\mathrm{cm\,s^{-2}}$, about 0.2% of the acceleration due to gravity. Its components are

$$a_{\mathrm{c},z} = a_\mathrm{c} \cos b = 1.54\,\mathrm{cm\,s^{-2}}, \tag{2}$$

along the vertical (upward) direction and

$$a_\mathrm{c,s} = a_\mathrm{c} \sin b = 1.68\,\mathrm{cm\,s^{-2}}, \tag{3}$$

along the southward direction.

2. — In equilibrium, the plumb bob must be parallel to the vector sum of the centrifugal and gravitational forces. If the inertial and gravitational masses are equal, the deviation from the radial direction is

$$\theta_1 = \tan^{-1}\left[\frac{a_\mathrm{c,s}}{g - a_{\mathrm{c},z}}\right] \tag{4}$$

which is about 0.098°.

3. — If the inertial and gravitational masses are not equal, the deviation becomes

$$\theta_2 = \tan^{-1}\left[\frac{a_\mathrm{c,s}}{(m_\mathrm{G}/m_\mathrm{I})g - a_{\mathrm{c},z}}.\right] \tag{5}$$

Let $m_\mathrm{G}/m_\mathrm{I} = 1 - \epsilon$ be the ratio of gravitational to inertial mass for the second line, where $\epsilon = 10^{-9}$. Using the small-angle approximations for $\tan\theta_1$ and $\tan\theta_2$, we find that the difference in the angles is

$$\frac{\Delta\theta}{\theta_1} \approx -\epsilon\,\frac{g}{(1-\epsilon)g - a_{\mathrm{c},z}} \approx -\epsilon. \tag{6}$$

That is, the angles differ by about one part in a billion.