Astrophysics — ASTR-4220
Class 12
Ionization Equilibrium

Background

In this exercise you will estimate the fractional ionization of hydrogen in the atmospheres of normal stars, assuming for simplicity that the plasma is pure hydrogen. We showed in class that

$$x^2 + \alpha x - \alpha = 0, \tag{1}$$

where

$$x \equiv \frac{n\left(\mathrm{H}^+\right)}{n_\mathrm{H}} \tag{2}$$

is the fractional ionization, n_H is the total number density of hydrogen nuclei, and α is a parameter with

$$\log \alpha \approx 21.68 - \log\left(n_\mathrm{H}/\mathrm{m}^{-3}\right) + \frac{3}{2}\log\left(T/\mathrm{K}\right) - \frac{68,600\,\mathrm{K}}{T}. \tag{3}$$

Given the density and temperature of an atmosphere, one computes α and then solves eq. (1) for x.

Exercise (35 pts)

Please give all answers clearly on the numbered answer sheet. You may attach your work on a separate sheet but it is not necessary.

1. (5 pts) — Write down the solution of eq. (1) for $x(\alpha)$. If there are multiple solutions, indicate which one is physically relevant.

2. (5 pts) — Based on simple physical considerations (i.e., not math), what is the limit of x for $\alpha \ll 1$? What is the limit for $\alpha \gg 1$?

3. (10 pts) — An attempt to evaluate the solution numerically (e.g., on a calculator) may fail badly if $\alpha \gg 1$, i.e., the solution in part 1 will *not* give you the limit in part 2. Find an approximation to the "formal" solution in part 1 that gives the right limit for $\alpha \gg 1$. Hint: Set $x = 1 - \epsilon$ where ϵ is a small number and find a well-behaved solution for $\epsilon(\alpha)$.

4. (10 pts) — Find an approximate but highly accurate solution when $\alpha \ll 1$.

5. (5 pts) — Now fill in the table for the fractional ionizations in the atmospheres of main sequence stars. Fill in $\log \epsilon$ if x is very close to unity; otherwise fill in $\log x$.

Solution

1. (5 pts) — The sacred quadratic formula gives

$$x = \frac{1}{2}\left[-\alpha + \left(\alpha^2 + 4\alpha\right)^{1/2}\right]. \tag{4}$$

2. (5 pts) — The limits are

$$\alpha \ll 1 \quad \longleftrightarrow \quad n \text{ large or } T \text{ small} \quad \longleftrightarrow \quad x \ll 1 \quad \text{(weakly ionized)} \tag{5}$$

and

$$\alpha \gg 1 \quad \longleftrightarrow \quad n \text{ small or } T \text{ large} \quad \longleftrightarrow \quad x \approx 1 \quad \text{(highly ionized).} \tag{6}$$

3. (10 pts) — Substituting $x = 1 - \epsilon$ into eq. (1) gives a quadratic equation for ϵ:

$$\epsilon^2 - (2 + \alpha)\,\epsilon + 1 = 0. \tag{7}$$

But $\epsilon^2 \ll \epsilon$ and $2 + \alpha \approx \alpha$ so the last expression is well approximated by

$$-\alpha\epsilon + 1 = 0. \tag{8}$$

The solution is

$$\epsilon \approx \alpha^{-1} \qquad \text{if } \alpha \gg 1 \tag{9}$$

or

$$\log \epsilon \approx -\log \alpha \qquad \text{if } \alpha \gg 1. \tag{10}$$

4. (10 pts) — For the weakly-ionized case ($\alpha \ll 1$) so $\alpha^2 \ll \alpha \ll \alpha^{1/2}$. Keeping only the term of order $\alpha^{1/2}$ on the RHS of (4) gives

$$x \approx \alpha^{1/2} \qquad \text{if } \alpha \ll 1 \tag{11}$$

or

$$\log x \approx \frac{1}{2}\log \alpha \qquad \text{if } \alpha \ll 1. \tag{12}$$

5. (5 pts) — See the attached table.

Spectral Type	$\log\left(n/\mathrm{m}^{-3}\right)$	$\log\left(T/\mathrm{K}\right)$	$\log\ \alpha$	$\log x$ OR $\log \epsilon$
O5	21.0	4.57	5.69	$\log \epsilon = -5.7$
B5	21.0	4.14	1.92	$\log \epsilon = -1.9$
A5	21.6	3.88	-3.14	$\log x = -1.6$
F5	22.6	3.77	-6.92	$\log x = -3.5$
G5	23.0	3.70	-9.46	$\log x = -4.8$
K5	23.4	3.58	-14.39	$\log x = -7.2$
M5	23.7	3.40	-24.23	$\log x = -12.1$

The table above gives typical physical properties in the photospheres of ordinary (main sequence) stars, where n is the total number density of atoms and T is the temperature. Assume for simplicity that the gas is pure hydrogen.