Astrophysics — ASTR-4220
Class 11
Complete Degeneracy

Background

Consider a gas of completely degenerate electrons. It is possible to rewrite the integral for the pressure in the form

$$P = \frac{8\pi}{3}\,\frac{m_{\rm e}^4 c^5}{h^3} \int_0^{x_{\rm F}} \frac{x^4 dx}{(1+x^2)^{1/2}} \;\equiv\; A\, f\left(x_{\rm F}\right), \tag{1}$$

where

$$x_{\rm F} \equiv \frac{p_{\rm F}}{m_{\rm e} c} \tag{2}$$

is the dimensionless Fermi momentum,

$$A = \frac{\pi}{3}\left(\frac{h}{m_{\rm e} c}\right)^{-3} m_{\rm e} c^2, \tag{3}$$

and f is dimensionless. Similarly, the energy density can be written

$$\frac{E}{V} = 8\pi\left(\frac{h}{m_{\rm e} c}\right)^{-3} m_{\rm e} c^2 \int_0^{x_{\rm F}} x^2 \left[\left(1+x^2\right)^{1/2} - 1\right] dx \;=\; A\, g\left(x_{\rm F}\right), \tag{4}$$

where g is dimensionless.

Exercise

1. (5 pts) — Show that the RHS of expression (1) has the correct units.

2. (5 pts) — Show that pressure and energy density have the *same* units. This establishes that expression (4) is also dimensionally correct.

3. (5 pts) — On intuitive grounds we expect A to be a typical value of P (or E/V). What is the value of A is SI units? What is A in atmospheres? Assume that the gas is made of electrons. (1 N/m^2 = 1 Pascal $\approx 9.87 \times 10^{-6}$ atm).

4. (5 pts) — Compare A to the central pressure in a white dwarf. Consider a typical white dwarf with mass $M = 0.5\, M_\odot$ and a radius equal to the Earth's radius.

5. (5 pts) — If $x_{\rm F} \ll 1$, is the gas nonrelativistic or extremely relativistic?

6. (5 pts) — For what range of the electron density, $n_{\rm e}$, is $x_{\rm F} \ll 1$?

7. (10 pts) — For small x, the Taylor series expansions of $f(x)$ and $g(x)$ are

$$f(x) \approx \frac{8}{5}x^5 - \frac{4}{7}x^7 + \ldots \tag{5}$$

and

$$g(x) \approx \frac{12}{5}x^5 - \frac{3}{7}x^7 + \ldots \tag{6}$$

Find approximate expressions for P and E when $x_\mathrm{F} \ll 1$. Your answers should be simple functions of x_F.

8. (5 pts) — Is the equation of state a gamma law when $x_\mathrm{F} \ll 1$? If so, what is γ?

9. (5 pts) For the range of densities you found in part 6, is the star stable or unstable?

Solution

1. Let $[Y]$ denote the units of quantity Y and let L denote length. Analyzing the LHS of expression (1) we find

$$\begin{aligned} [P] &= \left[\tfrac{\mathrm{Force}}{L^2}\right] \\ &= \left[\tfrac{\mathrm{force}\times L}{L^3}\right] \\ &= \tfrac{\mathrm{energy}}{\mathrm{volume}} \end{aligned} \tag{7}$$

Analyzing the RHS gives

$$\begin{aligned} [A] &= \left[\left(\tfrac{h}{m_\mathrm{e}c}\right)^{-3} m_\mathrm{e}c^2\right] \\ &= \left[L^{-3} \times \mathrm{energy}\right] \quad , \\ &= \tfrac{\mathrm{energy}}{\mathrm{volume}} \end{aligned} \tag{8}$$

where I used the fact that $h/m_\mathrm{e}c$ is the Compton wavelength (a length) and $m_\mathrm{e}c^2$ is an energy.

2. See previous answer.

3. The value is

$$A = 6.0 \times 10^{21}\,\mathrm{Pa} = 5.9 \times 10^{16}\,\mathrm{atm}. \tag{9}$$

4. To order of magnitude, the central pressure is

$$P \sim \frac{GM^2}{R^4} = 4.0 \times 10^{22}\,\mathrm{Pa} = 3.9 \times 10^{17}\,\mathrm{atm}. \tag{10}$$

5. If $x_\mathrm{F} \ll 1$ then $p_\mathrm{F} \ll m_\mathrm{e}c$. The gas is nonrelativistic.

6. Write x_F in terms of the density:

$$x_\mathrm{F} = \frac{p_\mathrm{F}}{m_\mathrm{e} c} = \left(\frac{3}{8\pi}\right)^{1/3} \left(\frac{h}{m_\mathrm{e} c}\right) n_\mathrm{e}^{1/3}. \tag{11}$$

Then $x_\mathrm{F} \ll 1$ implies

$$n_\mathrm{e} \ll \frac{8\pi}{3} \left(\frac{h}{m_\mathrm{e} c}\right)^{-3} = 6 \times 10^{35}\,\mathrm{m}^{-3}. \tag{12}$$

7. Keeping only the largest terms in the Taylor expansions gives

$$P \approx \frac{8}{5} A\, x_\mathrm{F}^5 \tag{13}$$

and

$$E/V \approx \frac{12}{5} A\, x_\mathrm{F}^5. \tag{14}$$

8. The ratio

$$\frac{P}{E/V} = \frac{2}{3}. \tag{15}$$

Since the ratio is constant, it is indeed a gamma law. The RHS of the last expression is $\gamma - 1$, so $\gamma = 5/3$.

9. Since $\gamma > 4/3$ the star is *stable*.