

# Physics I: A Mathematical Review

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## 1 Introduction

This document is a review of fundamental mathematical concepts for Physics I students at Rensselaer Polytechnic Institute. We will cover topics from algebra, trigonometry, vector analysis, and graphing that will be most useful during the course. For an in-depth review of these topics, please refer to the list of references at the end of the document.[1]

## 2 Algebra

### 2.1 Polynomials

We will begin with polynomials. A *polynomial* is a linear combination of algebraic expressions of any order. For example, the following is a polynomial of  $n^{\text{th}}$  order in  $x$

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{n-1}x^{n-1} + a_nx^n. \quad (1)$$

Here, the  $a_0, a_1, \dots, a_n$  are coefficients to the variable  $x$ . These polynomials are used to construct algebraic equations. An *algebraic equation* is a mathematical statement equating two algebraic expressions. For example, the following is an algebraic equation of the  $2^{\text{nd}}$  order in  $y$  and  $1^{\text{st}}$  order in  $x$ :

$$xy^2 + y = 2 + x \quad (2)$$

### 2.2 Factoring

Algebraic equations can be simplified and manipulated before they are solved. There are several techniques for doing this, but we will mostly be using the technique of *factoring*. The factoring technique involves the representation

of a polynomial expression as a product of irreducible polynomials. For example,

$$3y^5 + 9y^2 = 3y^2(y^3 + 3). \quad (3)$$

In general, one usually encounters factoring of a quadratic or  $2^{nd}$  order polynomials of the form

$$ax^2 + bx + c. \quad (4)$$

The general method of factoring the above expression is a two step process:

1. Find two coefficients  $b_1$  and  $b_2$  such that  $b_1 + b_2 = b$  and  $b_1 \times b_2 = ac$ .
2. If two coefficients  $b_1$  and  $b_2$  are found satisfying the above criteria, then the factorized quadratic expression is given by

$$\left(\frac{a}{b_2}x + 1\right)(b_2x + c). \quad (5)$$

Of course, when the above method doesn't quite work because you have complicated coefficients then one may employ the method of *completing the square*. This is achieved by rewriting expression (4) as follows

$$a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \quad (6)$$

where we can define  $B \equiv b/2a$  and  $C \equiv c - b^2/4a$ . So, we have

$$a(x + B)^2 + C. \quad (7)$$

Alternatively, you may use the *quadratic formula* to obtain the roots of the polynomial equation and use them as factors. Please refer to the section on *Solving Quadratic Equations* for this method.

There are several special identities that one can use for factoring certain quadratic and cubic polynomials. Refreshing one's memories on factoring of special  $2^{nd}$  and  $3^{rd}$  order polynomials:

$$\begin{aligned} x^2 + 2xy + y^2 &= (x + y)^2 \\ x^2 - 2xy + y^2 &= (x - y)^2 \\ x^2 - y^2 &= (x + y)(x - y) \\ x^3 + y^3 &= (x + y)(x^2 - xy + y^2) \\ x^3 - y^3 &= (x - y)(x^2 + xy + y^2). \end{aligned} \quad (8)$$

## 2.3 Solving Quadratic Equations

Most frequently you will be required to solve a quadratic equation during the course of your technical education. There are several methods available to solve quadratic equations, such as, factorization and the quadratic formula. You will be solving equations of the following nature

$$ax^2 + bx + c = 0. \quad (9)$$

### 2.3.1 Factorization Method

One of the methods is to factor the quadratic expression and the factors will yield the solutions. For example, a certain equation has the following factors

$$(x - \alpha)(x - \beta) = 0. \quad (10)$$

Then, the solutions are

$$\begin{aligned} x &= \alpha \\ x &= \beta. \end{aligned} \quad (11)$$

### 2.3.2 The Quadratic Formula

There exists a formula to determine the roots of the quadratic equation. By *completing the square* for the quadratic equation, one can obtain the quadratic formula:

$$x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad (12)$$

where  $x_{\pm}$  are the roots of the polynomial equation. This method can also be used to determine the factors of a second order polynomial

$$(x - x_+)(x - x_-) = 0. \quad (13)$$

## 2.4 Solving System of Equations

Essentially, solving a system of two equations boils down to two methods: *substitution method* and *elimination method*. The following is the general form of the two equations you will be solving for:

$$\begin{aligned} \alpha x + \beta y &= \gamma \\ \delta x + \eta y &= \tau, \end{aligned} \quad (14)$$

where we will be solving for the variables  $x$  and  $y$ . The first step in either method is to simplify the expressions into irreducible forms and then proceed.

### 2.4.1 Substitution Method

In this method, we use one equation to solve for one variable in terms of the other, and substitute that expression into the second equation to obtain the solutions.[2]

So, first we can solve for  $x$  from the first equation

$$x = \frac{\gamma - \beta y}{\alpha} \quad (15)$$

and place its value into the second equation

$$\delta \left( \frac{\gamma - \beta y}{\alpha} \right) + \eta y = \tau$$

to obtain the solution for  $y$

$$y = \frac{\tau\alpha - \delta\gamma}{\eta\alpha - \delta\beta}. \quad (16)$$

Here we have assumed that  $\alpha \neq 0$  and that  $\eta\alpha \neq \delta\beta$ . Hence, we obtain  $x$ :

$$x = \frac{\gamma}{\alpha} - \frac{\beta}{\alpha} \left( \frac{\tau\alpha - \delta\gamma}{\eta\alpha - \delta\beta} \right) = \frac{\beta\tau - \eta\gamma}{\beta\delta - \alpha\eta}. \quad (17)$$

This process can be really beneficial when one has a large equation where most of the expressions are constants.

### 2.4.2 Elimination Method

In this method, we try to eliminate one of the variables in one of the equations by adding or subtracting the two equations. So, for our system, we can multiply the first equation by  $-\frac{\eta}{\beta}$  and add the two equations to obtain an equation for  $x$ . After multiplication, we get

$$\begin{aligned} \frac{-\alpha\eta}{\beta}x - \eta y &= -\frac{\eta\gamma}{\beta} \\ \delta x + \eta y &= \tau. \end{aligned} \quad (18)$$

After addition of the two equations above, one obtains

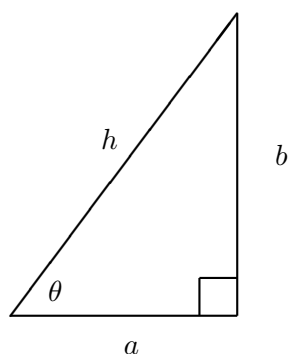
$$\begin{aligned} x \left( \delta - \frac{\alpha\eta}{\beta} \right) &= \tau - \frac{\eta\gamma}{\beta} \\ \text{i.e., } x &= \frac{\beta\tau - \eta\gamma}{\beta\delta - \alpha\eta}. \end{aligned} \quad (19)$$

Now inserting the above expression into our second equation, we obtain  $y$ :

$$y = \frac{\tau\alpha - \delta\gamma}{\eta\alpha - \delta\beta}. \quad (20)$$

These are the rudimentary algebraic techniques that one will mostly employ in the course.

### 3 Trigonometry



In trigonometry, we deal with the representation of the angles of a right triangle as a ratio of the length of its sides. We will cover trigonometric functions and some of the identities involved with trigonometry. The following are the basic trigonometric function definitions.

$$\begin{aligned} \sin \theta &= \frac{b}{h} & \csc \theta &= \frac{1}{\sin \theta} = \frac{h}{b} \\ \cos \theta &= \frac{a}{h} & \sec \theta &= \frac{1}{\cos \theta} = \frac{h}{a} \\ \tan \theta &= \frac{b}{a} & \cot \theta &= \frac{1}{\tan \theta} = \frac{a}{b}. \end{aligned} \quad (21)$$

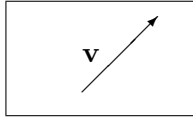
From these basic definitions, and the Pythagorean Theorem, one can obtain certain trigonometric identities:

$$\begin{aligned} \cos^2 \theta + \sin^2 \theta &= 1 \\ \tan^2 \theta + 1 &= \sec^2 \theta \\ 1 + \cot^2 \theta &= \csc^2 \theta \end{aligned} \quad (22)$$

These basic function definitions and identities are all you really need to know for this course.

### 4 Vectors

Simply stating, a vector is an object that has a magnitude and a direction in a given coordinate system. It is usually represented by the coordinate system in use, *e.g.*, in three dimensional Cartesian space a vector  $\mathbf{v} = (v_x, v_y, v_z)$ . A vector of unit length is known as a unit vector and is denoted by  $\hat{\mathbf{x}}$ .



The magnitude of a vector in three dimensions can be determined by the following method

$$v = \|\mathbf{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}. \quad (23)$$

Its direction with respect to the positive  $x$ -axis, in the  $x$ - $y$  plane, can be obtained by the following general equation

$$\theta = \arctan \frac{v_y}{v_x}. \quad (24)$$

When finding the angle of a vector, care must be taken to place it in the correct quadrant to insure that the original  $x$  and  $y$  components are recovered with correct signs:  $v_x = v \cos \theta$  and  $v_y = v \sin \theta$ . Normally, angles are given in the range  $(-180, 180]$  for degrees or  $(-\pi, \pi]$  for radians. This can be accomplished by taking the following steps:

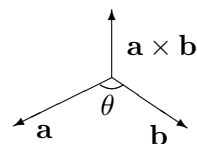
1. If  $v_x = 0$ , the angle is  $+90^\circ(+\pi/2)$  for  $v_y > 0$  or  $-90^\circ(-\pi/2)$  for  $v_y < 0$ . If so, then one is done.
2. Find  $\theta$  using equation (24) making sure to use the correct signs for both  $v_y$  and  $v_x$ . If  $v_x > 0$  then one is done as  $\theta$  lies in either quadrant I or IV.
3. If  $v_y \geq 0$  and  $v_x < 0$ , then add  $180^\circ(\pi)$  and the angle is placed in quadrant II.
4. If  $v_y < 0$  and  $v_x < 0$ , then subtract  $180^\circ(\pi)$  and the angle is placed in quadrant III.

To add or subtract two vectors, simply add or subtract its components in the corresponding dimension. For example, in two dimensions, to add two vectors  $\mathbf{a}$  and  $\mathbf{b}$  and obtain the resultant vector  $\mathbf{c}$ ,

$$\begin{aligned} c_x &= a_x + b_x \\ c_y &= a_y + b_y \\ \mathbf{c} &= (c_x, c_y). \end{aligned} \quad (25)$$

Its magnitude and direction can be obtained by using the equations mentioned above.

There are two special operations that are found to be quite useful in physics, *viz.*, *dot product* (or *scalar product*) and *cross product* (or *vector product*) of two vectors. The dot product of two vectors is defined as follows



$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos \theta = a_x b_x + a_y b_y + a_z b_z \quad (26)$$

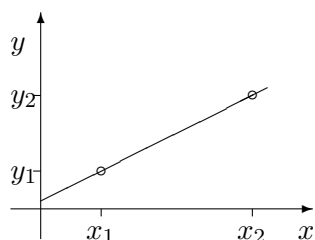
where  $\theta$  is the angle defined between the two vectors  $\mathbf{a}$  and  $\mathbf{b}$ . Please note that the result of this operation is a scalar, thus termed the *scalar product*. The cross product of two vectors is defined as follows

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - b_y a_z) \hat{\mathbf{x}} + (a_z b_x - b_z a_x) \hat{\mathbf{y}} + (a_x b_y - b_x a_y) \hat{\mathbf{z}} \quad (27)$$

Again, note that the result of this operation is a vector and the operation is thus named a *vector product*. For further reading in vector analysis, please refer to chapter 3 in your textbook.[4]

## 5 Graphing and Some Calculus<sup>1</sup>

In this course, we will use simple graphing techniques such as finding the slope of a curve, and the area under a curve. The slope of a linear curve



$$f(x) = mx + b \quad (28)$$

is given by  $m$ , which can be determined by the following expression associated with the two points labeled in the graph:

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \quad (29)$$

**Calculus:** You may recall that the derivative of the function  $f(x)$  yields the slope of the curve, *i.e.*,

$$\frac{df(x)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \quad (30)$$

<sup>1</sup>The calculus discussed here is meant to supplement the graphical review and may be omitted at the student's discretion.

You may also need to determine the area under a linear curve. The curve may be broken into rectangles, triangles and trapezoids, and the net area may then be computed. Recalling geometry,

$$A_{rectangle} = length \times width \quad (31)$$

$$A_{triangle} = \frac{1}{2}base \times height \quad (32)$$

$$A_{trapezoid} = \frac{1}{2}height \times (base_1 + base_2). \quad (33)$$

**Calculus:** Recall that the integral of a function  $f(x)$  with respect to its arguments over a finite interval gives the area under the curve, *i.e.*,

$$Area = \int_a^b f(x) dx = F(b) - F(a) \quad (34)$$

where  $F(x)$  is the anti-derivative of  $f(x)$ . [1]

## References

- [1] Weisstein, Eric W. From MathWorld—A Wolfram Web Resource. <http://mathworld.wolfram.com>
- [2] De Silva, Suranthe and D'Andreti, Peter. *Algebra* From Think Quest. 1997 <http://library.thinkquest.org>
- [3] Feynman, Richard, et al. *The Feynman Lectures on Physics: Volume I*. Reading: Addison-Wesley, 1963.
- [4] Halliday, David, et al. *Fundamentals of Physics: 7th Edition*. Hoboken: John Wiley & Sons, 2005.

## 6 Review Problems

### 6.1 Factoring

Factor the following polynomial expressions (0.5 points each):

1.  $8x^2 + 3y^2 + 10xy$
2.  $3x^2 - 9$
3.  $x^2 + 8x + 16$
4.  $3x^2 + 2x - 1$  (use complete the square)

### 6.2 Solving Equations

Solve the following equations for the required variables (0.5 points each):

1. Solve  $a^2x^2y - by^3x^2 - cb^2x + aby = 0$  for  $c$
2. Solve for  $x$  using the quadratic formula  $5x^2 + 2x - 3 = 0$
3. Solve for  $x$ :  $xy^2 - 3x + 8xy + 4y = 0$
4. Solve for  $x$  using the factor method:  $4x^2 - 8x + 4 = 0$
5. Solve for  $x$ :  $ay - bcx^2y = 0$

### 6.3 Solve System of Equations

Solve the first problem using the elimination method, and the next using the substitution method. (0.5 points each)

1.  
$$2x + 3y = -1 \quad \text{and} \quad 5x - 4y = 0$$
2.  
$$14x - 51y = -9 \quad \text{and} \quad -5x - 10y = 10$$

### 6.4 Trigonometry

Solve the following problems on Trigonometry (0.25 points each):

1. If  $\cos \theta = 1/2$ , then find  $\sin \theta$  and  $\tan \theta$ .
2. Calculate values of cosine, sine, and tangent for  $\theta = -\pi/4$ .

## 6.5 Vectors

1. Given a vector  $\mathbf{b}$  of magnitude 10 units and direction of 60 degrees from the x-axis, determine its x and y components. (0.5 points)
2. Given a vector's components  $c_x$  and  $c_y$ , determine its magnitude and direction. Please provide all angles between  $(-\pi, \pi]$  or  $(-180^\circ, 180^\circ]$ . (0.5 points each)
  - (a)  $c_x = 15$  units and  $c_y = 10$  units.
  - (b)  $c_x = -32$  units and  $c_y = 27$  units.
  - (c)  $c_x = 14$  units and  $c_y = -16$  units.
  - (d)  $c_x = -3$  units and  $c_y = -8$  units.
3. Given vector  $\mathbf{v}_1$  with magnitude 20 units in direction  $3\pi/4$  radians and  $\mathbf{v}_2$  with magnitude 15 units and angle  $\pi/3$  radians, determine the resultant vector  $\mathbf{v}$ . (0.5 points)

## 6.6 Graphing

Given the graph of  $f(x)$  below, graph the slope of this curve as a function of  $x$  and graph the area under this curve as a function of  $x$ . (1 points)

