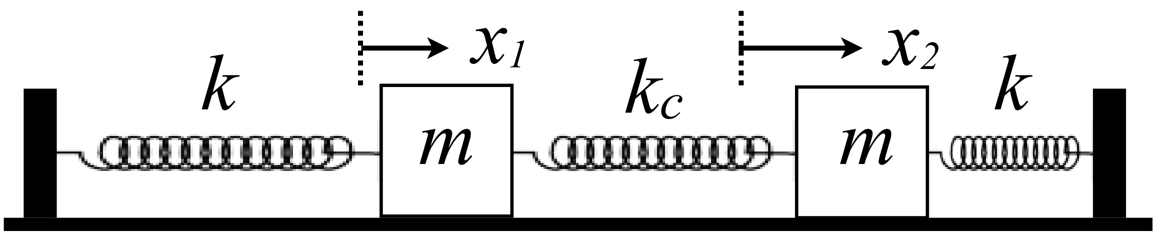
Second Midterm Exam is This Thursday!

The topic we call "coupled oscillations" has far reaching implications. The formalism ends up being appropriate for many different applications, some of which bear only a passing resemblance to classical oscillation phenomena. This includes the mathematics of eigenvalues and eigenvectors, for example.

These notes describe the elementary features of coupled mechanical oscillations. We use one specific example, and describe the method of solution and the physical implications implied by that solution. More complicated examples, and their solutions, are easy to come by, for example on the web.[1]

The following figure describes the problem we will solve:



Two equal masses m slide horizontally on a frictionless surface. Each is attached to a fixed point by a spring of spring constant k. They are "coupled" to each other by a spring with spring constant k_c. The positions of the two masses, relative to their equilibrium position, are given by x_1 and x_2 respectively.

Now realize an important point. We have *two* masses, each described by their own position coordinate. That means we will have *two* equations of motion, one in terms of $\ddot{x}_1$ and the other in terms $\ddot{x}_2$. Furthermore, since the motion of one of the masses determines the extent to which the spring k_c is stretched, and therefore affects the motion of the other mass, these two equations will be "coupled" as well. We will have to develop some new mathematics in order to solve these coupled differential equations.

We get the equations of motion from "$F = ma$", so let's do that first for the mass on the left, i.e., the one whose position is specified by x_1. It is acted on by two forces, the spring k on its left and the spring k_c on its right. The force from the spring on the left is easy. It is just $-kx_1$.

The spring on the right is a little trickier. It will be proportional to $(x_2 - x_1)$ since that is the extent to which the spring is stretched. (In other words, if $x_1 = x_2$, then the length of spring k_c is not changed from its equilibrium value.) It will also be multiplied by k_c, but we need to get the sign right. Note that if $x_2 > x_1$, then the string is stretched, and the force on the mass will be to the right, i.e. positive. On the other hand, if $x_2 < x_1$, then the spring is compressed and the force on the mass will be negative. This makes it clear that we should write the force on the mass as $+k_c(x_2 - x_1)$.

The equation of motion for the first mass is therefore

$$-kx_1 + k_c(x_2 - x_1) = m\ddot{x}_1$$

[1]See http://math.fullerton.edu/mathews/n2003/SpringMassMod.html.

We get some extra reassurance that we got the sign right on the second force, because if $k = k_c$, then the term proportional to x_1 does not cancel out.

The equation of motion for the second mass is now easy to get. Once again, from the spring k on the right, if x_2 is positive, then the spring pushes back so the force is $-kx_2$. The force from the "coupling" spring is the same magnitude as for the first mass, but in the opposite direction, so $-k_c(x_2 - x_1)$. This equation of motion is therefore

$$-kx_2 - k_c(x_2 - x_1) = m\ddot{x}_2$$

So now, we can write these two equations together, with a little bit of rearrangement:

$$\begin{aligned} (k + k_c)x_1 - k_c x_2 &= -m\ddot{x}_1 \\ -k_c x_1 + (k + k_c)x_2 &= -m\ddot{x}_2 \end{aligned}$$

To use some jargon, these are coupled, linear, differential equations. To "solve" these equations is to find functions $x_1(t)$ and $x_2(t)$ which simultaneous satisfy both of them. We can do that pretty easily using the exponential form of sines and cosines, which we discussed earlier when we did oscillations with one mass and one spring. Following our noses, we write

$$x_1(t) = a_1 e^{i\omega t} \qquad x_2(t) = a_2 e^{i\omega t}$$

where the task is now to see if we can find expressions for a_1, a_2, and ω which satisfy the differential equations. You'll recall that taking the derivative twice of functions like this, brings down a factor of $i\omega$ twice, so a factor of $-\omega^2$. Therefore, plugging these functions into our coupled differential equations gives us the *algebraic* equations[2]

$$\begin{aligned} (k + k_c)a_1 - k_c a_2 &= m\omega^2 a_1 \\ -k_c a_1 + (k + k_c)a_2 &= m\omega^2 a_2 \end{aligned}$$

This is good. Algebraic equations are a lot easier to solve than differential equations. To make things even simpler, let's divide through by m, do a little more rearranging, and define two new quantities $\omega_0^2 \equiv k/m$ and $\omega_c^2 \equiv k_c/m$. Our equations now become

$$(\omega_0^2 + \omega_c^2 - \omega^2)a_1 - \omega_c^2 a_2 = 0 \qquad (29)$$

$$-\omega_c^2 a_1 + (\omega_0^2 + \omega_c^2 - \omega^2)a_2 = 0 \qquad (30)$$

So, what have we accomplished? We think that some kind of conditions on a_1, a_2, and ω will solve these equations. Actually, we can see a solution right away. In mathematician's language, these are two coupled homogeneous (i.e. $= 0$) equations in the two unknowns a_1 and a_2. The solution has to be $a_1 = a_2 = 0$. Yes, that is a solution, but it is a very boring one. All it means as that the two masses don't ever move.

The way out of this dilemma is to turn these two equations into one equation. In other words, if the left hand side of one equation was a multiple of the other, then both equations would be saying the same thing, and we could solve for a relationship between a_1 and a_2,

[2] A mathematician would call collectively call these two equations an eigenvalue equation. The reason becomes clearer when you write this using matrices.

but not a_1 and a_2 separately. Mathematically, this condition is just that the ratio of the coefficients of a_2 and a_1 for one of the equations is the same as for the other, namely[3]

$$\begin{aligned}
\frac{-\omega_c^2}{\omega_0^2+\omega_c^2-\omega^2} &= \frac{\omega_0^2+\omega_c^2-\omega^2}{-\omega_c^2} \\
\omega_c^4 &= (\omega_0^2+\omega_c^2-\omega^2)^2 \\
\pm\omega_c^2 &= \omega_0^2+\omega_c^2-\omega^2 \\
\omega^2 &= \omega_0^2+\omega_c^2\mp\omega_c^2
\end{aligned}$$

In other words, these two equations are really one equation if

$$\omega^2 = \omega_0^2 \equiv \omega_A^2$$

or, instead, if

$$\omega^2 = \omega_0^2 + 2\omega_c^2 \equiv \omega_B^2$$

Borrowing some language from the mathematicians, the physicist refers to ω_A^2 and ω_B^2 as "eigenvalues." We will also refer to A and B as "eigenmodes."

What is the physical interpretation of the eigenmodes? To answer this, we go back to Eq. 29 or, equivalently, Eq. 30. (Remember, they are the same equation now.) If we substitute $\omega^2 = \omega_A^2$ into Eq. 29 we find that

$$a_1^A = a_2^A$$

where the superscript just marks the eigenmode that we're talking about. In other words, the two masses move together in "lock step", with the same motion. This happens when the frequency is $\omega = \pm\omega_0 = \pm\sqrt{k/m}$, and that is just what you expect. It is as if the two masses are actually one, with mass $2m$. The effective spring constant is $2k$, as you learned in your laboratory exercise. Therefore, we expect this double-mass to oscillate with $\omega^2 = (2k)/(2m) = k/m$. Good. This all makes sense.

Now consider eigemode B. Substituting $\omega^2 = \omega_0^2 + 2\omega_c^2$ into Eq. 29, we find that

$$a_1^B = -a_2^B$$

In other words, the two masses oscillate "against" each other, and with a somewhat higher frequency. In this case, it is interesting to see the difference between $\omega_c \ll \omega_0$ (in other words, a very weak coupling spring) and $\omega_c \gg \omega_0$ (strong coupling spring). For the weak coupling spring, the frequency is just the same as ω_0, and that makes sense. If the two masses aren't coupled to each other very strongly, then they act as two independent masses m each with their own spring constant k. On the other hand, for a strong coupling between the masses, the frequency is $\omega = \omega_c\sqrt{2}$, which gets arbitrarily large. Sometimes, this "high frequency mode" can be hard to observe.

A simple experimental test of this result is to set up two identical masses with three identical springs. In other words, $\omega_c = \omega_0$. In that case $\omega_B^2 = 3\omega_A^2$ and therefore the frequency for mode B should be $\sqrt{3}$ times larger than the frequency for mode A. We should be able to make this test as a demonstration in class.

We have been talking about general properties of the two eigenmodes. Of course, this doesn't tell you how the system behaves given certain starting values, that is, specific initial conditions.

[3] A mathematician would say that we are setting the determinant equal to zero.

For this, we need to understand that the two eigenmodes can be combined. In fact, the general motion of each of the masses really needs to be written as a sum of the two eigenmodes. Each of these comes with a positive and negative frequency, just as it did when we applied all this to the motion of a single mass and spring. (Recall that the sum and difference of a positive and negative frequency exponential, are equivalent to a cosine and sine of that frequency.) Incorporating what we've learned about the relative motions of modes A and B, we can write the motions as follows:

$$\begin{aligned} x_1(t) &= ae^{i\omega_A t} + be^{-i\omega_A t} + ce^{i\omega_B t} + de^{-i\omega_B t} \\ x_2(t) &= ae^{i\omega_A t} + be^{-i\omega_A t} - ce^{i\omega_B t} - de^{-i\omega_B t} \end{aligned}$$

where a, b, c, and d are constants which are determined from the initial conditions.

An obvious set of initial conditions is starting mass #1 from rest at $x = x_0$, and with mass x_2 from rest at its equilibrium position. This leads to the solution

$$x_1(t) = (x_0/2)\left[\cos(\omega_A t) + \cos(\omega_B t)\right] \tag{31}$$

$$x_2(t) = (x_0/2)\left[\cos(\omega_A t) - \cos(\omega_B t)\right] \tag{32}$$

Practice Exercise

1) Show that these equations satisfy the equations of motion *and* the initial conditions.

2) Make a plot of $x_1(t)$ and $x_2(t)$. Choose some value for x_0 and ω_A. Also choose a value for ω_B that represents a "weak coupling" system. Explain the motion of each of the two masses in terms of conservation of mechanical energy. It would be helpful if you rewrite Eqs. 31 and 32 using the trigonometric identities

$$\begin{aligned} \cos(u) + \cos(v) &= 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \\ \cos(u) - \cos(v) &= -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right) \end{aligned}$$