First class! Name and course on board, read roll, introduce TA’s.

Distribute outline, lecture, and lab schedules. Discuss them.

New material for today: Start “thinking like a physicist.” Two new ideas, namely

(1) Solving problems with dimensional analysis, and
(2) $F = ma$ is a “differential equation”

**Dimensional Analysis**

Given a problem, ask “What is the important physics?”. Then list the quantities that have something to do with that physics. In many cases, you can combine these things to get the answer, at least to within a factor of 2 or $\pi$ or something like that.

Convenient notation:
- $[\text{distance}] = L$
- $[\text{time}] = T$
- $[\text{mass}] = M$
- $[\text{temperature}] = K$

**Example: What is the radius of a black hole?** This is a combination of gravity and relativity, so use $M$, $G$, and $c$. Let the radius be $R = M^x G^y c^z$ where we need to find $x$, $y$, and $z$. Newton’s constant $G$ has units m$^3$/kg-sec$^2$, so $[G] = L^3 M^{-1} T^{-2}$. This gives

$$[R] = M^x L^{3y} M^{-y} T^{-2y} T^{z} T^{-z} = M^{x-y} L^{3y+z} T^{-2y-z} = L^1$$

so $x = y$, $3y + z = 1$, and $2y + z = 0$. The second and third of these say that $y = 1$, and so $x = 1$ and $z = -2$. Therefore we get

$$R = GM/c^2$$

The “correct answer” is $\frac{1}{2} GM/c^2$, called the Schwarzschild Radius.

(Hint: Useful website for units and constants is http://pdg.lbl.gov/2007/reviews/contents_sports.html#constantsetc.)

**Newton’s Second Law as an “Equation of Motion”**

The most important of Newton’s Laws is the second. For motion in one dimension, we write $F = ma$. (We’ll get to $\vec{F} = m\vec{a}$ later this week.) Physicists refer to this as the “equation of motion” for a particle moving through time in one dimension of space.

Let $x(t)$ be the position $x$ of a particle at time $t$. Then the velocity is $v(t) = dx/dt \equiv \dot{x}$. (Have you all seen this notation before??) Acceleration is $a(t) = dv/dt = \ddot{v} = \ddot{x}$. So Newton’s Second law can be written as

$$\ddot{x} = \frac{1}{m} F(x, t) \tag{1}$$

where we make it clear that the force $F$ may itself depend on position and/or time.

Equation 1 is called a “differential equation.” Unlike an algebraic equation, whose solution is a value for a quantity like, say, $x$, a the solution of a differential equation is a function, say $x(t)$. Since $x(t)$ describes the motion of a particle with mass $m$ acted upon by a force $F(x, t)$, we call Eq. 1 the “equation of motion.”
Simple example. Let \( F(x, t) = F_0 \), a constant independent of both \( x \) and \( t \). Then

\[
\begin{align*}
\ddot{x} &= \dot{v} = \frac{dv}{dt} = \frac{F_0}{m} \\
v(t) &= \frac{dx}{dt} = \frac{F_0 t + v_0}{m} \\
x(t) &= \frac{1}{2} \frac{F_0}{m} t^2 + v_0 t + x_0
\end{align*}
\]

where the last line is the solution to this equation of motion. This might look more familiar to you from your high school physics course, if we write \( a = F_0/m \). You probably called this “motion under constant acceleration”, but of course a constant acceleration implies a constant force, so it’s the same thing.

Each step in the integration we did involves a “constant of integration.” I called these constants \( v_0 \) for the first step, and \( x_0 \) for the second step. These are standard notations, and in fact good ones. The constant \( v_0 \) is the velocity at time \( t = 0 \), also called the “initial velocity.” Similarly, \( x_0 \) is the “initial position.” In order to solve any equation of motion, we will need to specify these “initial conditions.”

**Practice Exercises**

(1) Astronomers observe that distant galaxies recede from us at a nearly uniform rate. That is, a galaxy at a distance \( d \) recedes at a speed \( v = H d \), where \( H = 75 \text{ km/sec/Mpc} \) is called the Hubble Constant. (The “mega parsec” Mpc is a standard cosmological measure of length, equal to \( 3.1 \times 10^{22} \text{m} \).) What is the age of the universe? (You are welcome to make estimates instead of looking up exact values, but that is up to you. Did you know that the number of seconds in a year is very close to \( \pi \times 10^7 \)?)

(2) Let \( y(t) \) describe the one dimensional motion of a particle with mass \( m \) that moves vertically. Assume that the particle is acted on by a (constant) force \( F = -mg \). (Isn’t that peculiar, a force that is exactly proportional to the quantity \( m \) that appears in the equation of motion?) Assume that the particle starts out at \( y = 0 \) with an initial (upward) velocity \( V \). Find the maximum height to which the particle climbs, in terms of \( m \), \( g \), and \( V \). Make a sketch of \( y(t) \) as a function of \( t \). What is the shape of the curve? (Mathematicians have a name for it, and I’m sure you’ve heard it before.)
Today we will work with Newton’s Second Law in terms of vectors, i.e. \( \vec{F} = m\vec{a} = m\vec{\ddot{r}}. \)

First, however, let’s go over some preliminaries.

**Calculus Tools: Rules we will use**

I will try to review the calculus rules before we use them. Your math class will show you where these rules come from.

Last class used \( \frac{d}{dt} t^n = nt^{n-1} \) which means the same as \( \int t^n dt = \frac{t^{n+1}}{n+1} \). (We will worry about the case \( n = -1 \) later.) We also used \( \frac{d}{dt} af(t) = a \frac{df}{dt} \) where \( a \) is a constant.

This class: \( \frac{d}{dt} u[v(t)] = \frac{du}{dv} \frac{dv}{dt} \) “Chain Rule”; and \( \frac{d}{dt}(uv) = \frac{du}{dt}v + u \frac{dv}{dt} \) “Product Rule”

**Vectors (in 2D for now): See Textbook Appendix H**

Just scratching the surface of an important concept in physics, but right now you can think of vectors as “shorthand” for working in two (or three) directions at the same time.

Notation: \( \hat{i} \) (\( \hat{j} \)) is a “unit vector” in the \( x \) (\( y \)) direction. (\( \hat{k} \) for \( z \))

So, we write a vector \( \vec{a} = a_x \hat{i} + a_y \hat{j} \). The two values \( a_x \) and \( a_y \) determine the vector \( \vec{a} \).

[Draw a picture of a stick vector with components labeled, on \( x, y \) axes, like Fig.2-2.]

Examples:

\( \vec{r} = x\hat{i} + y\hat{j} \quad \vec{\dot{r}} = \vec{v} = v_x\hat{i} + v_y\hat{j} = \dot{x}\hat{i} + \dot{y}\hat{j} \)

The length of a vector \( \vec{a} \) is \( |\vec{a}| = (a_x^2 + a_y^2)^{1/2} \).

That’s enough for now. We’ll develop more concepts with vectors as we need them later.

**Newton’s Second Law (again)**

\( \vec{F} = m\vec{\ddot{r}} \) is really two separate differential equations:

\[
\begin{align*}
F_x &= m\ddot{x} \\
F_y &= m\ddot{y}
\end{align*}
\]

Solve for the motion by considering the equations separately. Hopefully that’s easy to do, but it depends on \( \vec{F} \). Sometimes \( F_x \) can depend on \( y \) and vice versa. That can make things hard, but it can also make them interesting. Maybe different coordinate systems will help.

The standard example is **projectile motion**, namely \( F_x = 0 \) and \( F_y = -mg \). (Section 4-3.)

The solutions of the two differential equations follow very simply from last class:

\[
\begin{align*}
x(t) &= x_0 + v_{x_0}t \\
y(t) &= y_0 + v_{y_0}t - \frac{1}{2}gt^2
\end{align*}
\]

So, we have solved the equation of motion. For any time \( t \) we now know where the particle will be located. It is at \( \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} \). Of course, to answer any question numerically, we need to specify the “initial conditions” \( \vec{r}_0 = x_0\hat{i} + y_0\hat{j} \) and \( \vec{v}_0 = v_{x_0}\hat{i} + v_{y_0}\hat{j} \).

So what path does the particle follow? Simplify by setting \( x_0 = y_0 = 0 \). That is, define the origin to be the place from where the “projectile” is launched. Then \( t = x/v_{x_0} \) and

\[
y = \frac{v_{y_0}}{v_{x_0}}x - \frac{1}{2} \frac{g}{v_{x_0}^2}x^2 \tag{2}
\]
The path is a parabola, which passes through the point \((x, y) = (0, 0)\). The “range” of the projectile is the other value of \(x\) for which \(y = 0\).

**Something new**

Let’s talk now about a special class of forces \(\vec{F}\). These are called “central forces” because they always point back to some central point.

![Diagram of a central force in two dimensions. The geometry of the rectangles makes it clear that \(F_y/F_x = y/x\), or equivalently \(xF_y - yF_x = 0\).](image)

Consider a new quantity which I will call \(\ell\), which is defined as follows:

\[
\ell \equiv m (x\dot{y} - y\dot{x})
\]  

(3)

Since \(x = x(t)\) and \(y = y(t)\) are functions of time, we also expect \(\ell = \ell(t)\). However,

\[
\frac{d\ell}{dt} = m (\dot{x}\dot{y} + x\ddot{y} - \dot{y}\ddot{x} - y\ddot{x}) = m (x\dot{y} - y\dot{x}) = xF_y - yF_x = 0
\]

The time derivative of \(\ell\) is zero. In other words, \(\ell\) does not change with time! This is your first example of “a constant of the motion.” We say that \(\ell\) is “conserved.” Our derivation shows that this is true for any central force.

Picture of stuff to come: \(\ell\) is called the “\(z\)-component of angular momentum.” Indeed, it is the \(z\)-component of the (three dimensional) vector \(\vec{L} \equiv \vec{r} \times (m\vec{v})\). We’ll explain cross products and angular momentum in more detail later.

**Practice Exercises**

(1) Find the range \(R\) of the projectile whose arc is given by Eq. 2. Also find the maximum height \(h\) which the projectile achieves. Try to express your answers in terms of the initial speed \(v_0 \equiv |\vec{v}_0|\) and the “launch angle” \(\phi = \tan^{-1}(v_{y0}/v_{x0})\).

(2) Consider a one-dimensional force \(F = -kx\). Show that \(E \equiv \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\) is a constant of the motion.
Hand in homework.

Pep talk: Don’t be scared!! Recommend Schaum’s Math Handbook

Today: Kinetic, potential, and total mechanical energy

First: Any questions on the labs, what we’re looking for, etc?

**Introduction:** Practice Exercise #2 from last Thursday, 30 August.

“For $F = -kx$, show that $E \equiv \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ is a constant of the motion.”

Go slowly: \[
\frac{dE}{dt} = m\ddot{x} + k\dot{x} = \ddot{x}(m\dot{x} + kx) = \dot{x}(ma - F) = 0
\]

So, $E$ is conserved. This $E$ just works for this $F(x) = -kx$, but it is obvious how we generalize this. Define $U(x)$ so that $dU/dx = -F(x)$. Now $E \equiv \frac{1}{2}m\dot{x}^2 + U(x)$. Then \[
\frac{dE}{dt} = m\ddot{x} + \frac{dU}{dt} = m\ddot{x} + \frac{dU}{dx}\frac{dx}{dt} = \dot{x}\left(m\ddot{x} + \frac{dU}{dx}\right) = \dot{x}(ma - F) = 0
\]

We call $\frac{1}{2}m\dot{x}^2 \equiv K$ the “kinetic energy.” $U(x)$ is the “potential energy”, and $E$ is the total mechanical energy. Only $E$ is conserved.

**The “Complete Solution” to motion in one dimension**

Emphasize that there is an alternative to “solving the equation of motion” if one discusses motion in terms of energy.

\[
E \equiv \frac{1}{2}m\dot{x}^2 + U(x)
\]

\[
\dot{x} = v = \pm \sqrt{\frac{2}{m}[E - U(x)]}
\]

Two ways to look at this:

a) Velocity can be to the left or right, but “speed” ($\equiv |v|$) is determined. The square root leads to turning points to the motion, which limit the position based on the energy.

Textbook Figure 12-8. Label $K$ abd $U$ at some position $x$. Relate force plot to the energy plot. Discuss behaviors of particles with various $E_i$, and mention oscillations. Draw the analogy with “up and down a hill.”

b) In principle, one can get the motion $x(t)$ by integrating Eq. 4. We’ll do this in a practice problem.
Work and Energy in one and two dimensions; Line integrals.

Traditionally we start talking about “work” before talking about “energy” because this has some practical use in engineering applications, and because it helps make the distinction between “conservative” forces, for which a potential can be defined, and “non-conservative” forces (like friction) for which it cannot.

Math concept: “Dot Product” (Textbook Appendix H-4)
\[ \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y = |\vec{A}| |\vec{B}| \cos \phi \]
where \( \phi \) is the angle between \( \vec{A} \) and \( \vec{B} \).

Note that \( \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = 1 \) and \( \hat{i} \cdot \hat{j} = 0 \) so this all works fine if we write \( \vec{A} = A_x \hat{i} + A_y \hat{j} \) and \( \vec{B} = B_x \hat{i} + B_y \hat{j} \) and just multiply.

Work is “force (\( F \)) applied through a distance (\( s \)).” Successive ways to write work \( W \):

| Constant force in one dimension | \( W = Fs \) |
| Variable force in 1D over small distance \( dx \) | \( dW = F(x)dx \) |
| Variable force in 1D between points \( a \) and \( b \) | \( W = \int_a^b dW = \int_a^b F(x)dx \) |
| Constant force at an angle \( \phi \) | \( W = Fs \cos \phi = \vec{F} \cdot \vec{s} \) |
| Variable force over a small distance \( d\vec{s} \) | \( dW = \vec{F}(x) \cdot d\vec{s} = F_x dx + F_y dy \) |
| Variable force between points \( a \) and \( b \) | \( W = \int \vec{F}(x) \cdot d\vec{s} \) (∗) |
| Variable force for a closed loop path | \( W = \oint \vec{F}(x) \cdot d\vec{s} \) (∗) |

The second relation makes it clear that \( U(x) = -W(x) \) but there is an important subtlety. If potential energy is to make sense, it cannot depend on the path one takes to get from point \( a \) to point \( b \). In other words, we can only define a potential energy \( U(x) \) for a force \( \vec{F}(x) \) with the property

\[ \oint \vec{F}(x) \cdot d\vec{s} = 0 \]

that is, the work done around a closed path must be zero. We call such forces “conservative” because we can come up with a \( U(x) \) for them and therefore talk about “conservation of mechanical energy.”

Friction (or drag or things like that) is the prototypical “non-conservative” force. The work done by a constant frictional force \( f \) is \( W = -fs \) regardless of the direction of travel. Clearly there will be nonzero (in fact negative) work done over a closed loop.

Practice Exercises

1. Use Eq. 4 with \( U(x) = mgx \) and \( E = mgh \) to find \( x(t) \). Assume that \( x = h \) when \( t = 0 \). (This describes the motion of something falling from rest at a height \( h \), right?) Split \( dx/dt \) so that you get \( \int dx \) on the left and \( \int dt \) on the right. It should be obvious after integrating that the constant of integration is zero for this initial condition. You’ll need the following obscure calculus rule:

\[ \int \frac{dx}{\sqrt{a + bx}} = \frac{2\sqrt{a + bx}}{b} \]

where \( a \) and \( b \) are constants.

2. Show that the force \( \vec{F}(x, y) = y \hat{j} \) is non-conservative, by integrating the work around a closed loop and showing that the result is non-zero. Pick a simple loop, for example, a rectangle parallel to the \( x \) and \( y \) axes, and with one corner at \((0, 0)\) and the other corner at \((a, b)\). Try the same thing for \( \vec{F}(x, y) = x \hat{i} \) and show that it is conservative.
Solution to Practice Problem 2

Problem

Show that the force \( \vec{F}(x, y) = y \hat{i} \) is non-conservative, by integrating the work around a closed loop and showing that the result is non-zero. Pick a simple loop, for example, a rectangle parallel to the \( x \) and \( y \) axes, and with one corner at \((0, 0)\) and the other corner at \((a, b)\). Try the same thing for \( \vec{F}(x, y) = x \hat{i} \) and show that it is conservative.

Solution

Divide the rectangular loop into four sections \( I, II, III, \) and \( IV \) as shown below.

We can therefore write the work around the closed loop as

\[
W = \oint \vec{F} \cdot d\vec{s} = \int_I \vec{F} \cdot d\vec{s} + \int_{II} \vec{F} \cdot d\vec{s} + \int_{III} \vec{F} \cdot d\vec{s} + \int_{IV} \vec{F} \cdot d\vec{s} \tag{5}
\]

and do the integrals one by one. This is a good idea because \( d\vec{s} \) is parallel to either \( x \) or \( y \) in each of these four sections.

First consider the force \( \vec{F}(x, y) = y \hat{i} \), that is \( F_x = y \) and \( F_y = 0 \). Along paths \( II \) and \( IV \) we have \( d\vec{s} = \pm dy \hat{j} \), respectively. Therefore \( \vec{F} \cdot d\vec{s} = \pm F_y dy = 0 \) along each of these two segments. Also, along path \( I \), \( d\vec{s} = +dx \hat{i} \) so \( \vec{F} \cdot d\vec{s} = F_x dx = ydx \). However, \( y = 0 \) all along path \( I \). Therefore, the integrals along lines \( I, II, \) and \( IV \) in Eq.5 are all zero.

On the other hand, along path \( III \), \( \vec{F} \cdot d\vec{s} = -F_x dx = -ydx = -bdx \). Hence

\[
\int_{III} \vec{F} \cdot d\vec{s} = \int_0^a (-b) dx = -ab 
\]

Therefore \( W \neq 0 \) around the closed path and \( \vec{F}(x, y) = y \hat{i} \) is non-conservative.

Now do \( \vec{F}(x, y) = x \hat{i} \). Once again, the integrals along lines \( II \) and \( IV \) are zero, however

\[
\int_I \vec{F} \cdot d\vec{s} = \int_0^a (+x) dx \quad \text{and} \quad \int_{III} \vec{F} \cdot d\vec{s} = \int_0^a (-x) dx
\]

You can do the integrals if you want (they are \( \pm a^2/2 \)) but it is already clear that they are the negative of each other, and will cancel in Eq.5. Therefore, for this force, the work done around the closed path is zero.

At some point in the future, you will learn about a mathematical concept called “curl” and something called Stoke’s Theorem. These will be more formal ways of evaluating these sorts of integrals around closed loops.
Some Recap: Energy, Work, and Line Integrals

Ask for questions re solution of Practice Problem #2 from Thursday.

Uniform Circular Motion

First let’s talk about how to describe the motion, i.e. “kinematics.” Then, we’ll talk about the forces that can bring about this motion, i.e. “dynamics.”

By “uniform circular motion” we mean motion in a circle with constant speed \( v \). The velocity, however, is constantly changing in direction. Since velocity is changing, the acceleration is nonzero, even though the speed is constant.

Demonstrate with the pink string and small mass. Where’s the force? Does it do work??

This is a good example of how to analyze a “derivative” in physical terms, as opposed to mathematical ones. Refer to the figure below. The acceleration (vector) is

\[
\vec{a} = \frac{d\vec{v}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \lim_{\theta \to 0} \frac{\vec{v}_2 - \vec{v}_1}{2\theta r/v} \quad \text{(Take it slowly!)}
\]

Textbook Figure 4-16, showing the geometry used to “physically” find the derivative \( \frac{d\vec{v}}{dt} \). The speed \( v \) of the particle moving in the circle is the magnitude of any velocity vector along the path. The circle radius is \( r \), and the path length subtended by any angle \( \phi \) (in radians) is just \( r\phi \).

The inset shows that \( \Delta \vec{v} \) (1) points towards the center of the circle, and (2) has magnitude \( 2 \times v \sin \theta \).

The figure shows that \( \vec{a} \) points towards the center of the circle. That’s important! The geometry also allows us to write

\[
a = |\vec{a}| = \lim_{\theta \to 0} \frac{2v \sin \theta}{2\theta r/v} = \frac{v^2}{r} \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \frac{v^2}{r}
\]

Later when we study Taylor Series and their applications, we’ll see that \( \sin \theta / \theta \to 1 \) as \( \theta \to 0 \), but it’s easy enough for you to check. Use your calculator to find \( \sin \theta \) for small values of \( \theta \) (in radians!). You’ll see that as \( \theta \) gets smaller and smaller, \( \sin \theta \) gets closer and closer to \( \theta \) itself.

Now that we know the kinematics of uniform circular motion, we can describe the physical processes behind anything moving in such a circle. Applications include motion on race tracks, tension in spinning mechanical systems, binary star behavior, and dark matter and black holes in galaxies, to name a very few.

Today, we’ll apply it to the “conical pendulum.”
One Application: The Conical Pendulum

Demonstrate it with the pink string and large mass. Emphasize how the behavior is very different from that of a “swinging” pendulum.

Talk a little about oscillations. Here’s an interesting case, where the oscillations have nothing to do with exchanging potential energy for kinetic energy. So, maybe oscillation is a bad name for this phenomenon? You say tomato, . . .

Our goal: Find the period $P$ of the “pendulum” in terms of any parameters that matter.

Textbook Figure 5-18. The mass $m$ executes uniform circular orbits with radius $R$. If the speed is $v$, then the period is just the time it takes to execute one orbit, namely $P = 2\pi R/v$. So, we have to figure out $v$ in terms of $R$, or perhaps in terms of the string length $L$ and the hanging angle $\theta$, since $R = L \sin \theta$.

We figure this out from the acceleration $a = v^2/R$, directed towards the center of the circle. Since we know that $F = ma$, we just need to figure out what the force $F$ is. Clearly, this force is provided by the tension $T$.

The component of the tension that is directed towards the center of the circle (i.e., in the direction of $\vec{a}$) is just $F = T \sin \theta$. But the upward component of the tension must balance gravity $mg$, so $mg = T \cos \theta$. This gives

$$F = mg \tan \theta = m \frac{v^2}{R} \quad \text{and} \quad v^2 = Rg \tan \theta$$

and so

$$P = \frac{2\pi R}{v} = 2\pi \sqrt{\frac{R}{g \tan \theta}} = \frac{2\pi}{g} \sqrt{\frac{L \cos \theta}{g}}$$

When we learn about the swinging pendulum, you’ll see that the period is (to a good approximation $2\pi \sqrt{L/g}$). It is interesting to note that the conical pendulum period approaches this value as the cone angle gets very small.

**Practice Exercises**

1. Ignore our derivation of “centripetal” acceleration. (That’s the name given to the acceleration towards the center in circular motion.) Instead, assume that something moves with a speed $v$ in a circle of radius $r$, and use dimensional analysis to make a guess at the acceleration. How does it compare to the right answer?

2. See Exercise 5-43. An object sitting on a horizontal surface will experience a force of “static friction” $f = \mu (mg)$ that resists horizontal motion. A small object sits on a turntable 13 cm from the center. When the turntable runs at 33-1/3 RPM (i.e. revolutions per minute), the object stays put. However, at 45 RPM the object flies off. Derive limits for the quantity $\mu$, called the “coefficient of static friction.” (You may have seen turntables which run at these speeds in museums or perhaps in your parents’ closets.)
Hand in homework.

Today: Newton’s philosophy, gravity, circular orbits, and “dark matter”

Mass! What a concept. Is it just \( F/a \) (“inertia”) or does it have a deeper meaning?
⇒ Gravity! Newton was brilliant! (Recommend Gleick’s book)

**Newton’s Law of Universal Gravitation**

“Two (point) masses \( m_1 \) and \( m_2 \), separated by a distance \( r \), attract each other along the line between them.” The magnitude of this force is

\[
F = G \frac{m_1 m_2}{r^2}
\]

or all together we write

\[
\vec{F} = -G \frac{m_1 m_2}{r^2} \hat{r}
\]

where \( G = 6.67 \times 10^{-11} \text{Nm}^2/\text{kg}^2 \) is called the **gravitational constant**, and was first measured by Henry Cavendish in 1798 with a torsion balance. See textbook Fig.14-5.

The earth is (nearly) a sphere with radius \( R_E \). For an object with mass \( m \) on the earth’s surface, use Shell Theorem (see below!) to get force from gravity. That is

\[
F = G \frac{m M_E}{R_E^2} = mg \quad \Rightarrow \quad g = G \frac{M_E}{R_E^2}
\]

Easy to see how this scales with planets of different size and density, since \( M = \rho \left( \frac{4}{3} \pi R^3 \right) \).

Note the old adage: “Cavendish weighted the Earth.”

**The Shell Theorems**

**Shell Theorem #1.** A uniformly dense spherical shell attracts an external particle as if all the mass of the shell were concentrated at its center.

**Shell Theorem #2.** A uniformly dense spherical shell exerts no gravitational force on a particle located anywhere inside it.

See your textbook for the proofs. Good example of intellectual usefulness of calculus!

**Gravitational Potential Energy**

Textbook Figure 14-11. Gravity is a conservative force. The work \( W = \int_{a}^{b} \vec{F} \cdot d\vec{s} \) done by gravity is independent of the path. No work is ever done on a circular arc, and the same work is done when changing the radius \( r \), regardless of the “angle”. Since I can make up any path by crawling along arcs and changing radii, my assertion is true. To find the potential energy due to gravity, just realize that \( F = -dU/dr \) where

\[
U(r) = -G \frac{m M}{r}
\]

for a mass \( m \) attracted to a mass \( M \) at a distance \( r \).
Classic application is escape velocity. Imagine launching something straight up from the surface of something with mass $M$ and radius $R$. How fast do I need to send it so that it barely “escapes”, i.e. makes it infinitely far away with zero velocity?

Total mechanical energy is conserved. Infinitely far away the gravitational potential energy is zero, and it is not moving so the kinetic energy is zero, too. So the total energy at the beginning is also zero:

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = 0 \quad \Rightarrow \quad v = \sqrt{\frac{2GM}{R}}$$

For earth, this is 11.2 km/sec. See Table 14-2 for other objects, especially “neutron star.”

**Circular Orbits and Dark Matter**

Just use our result from last week for “centripetal acceleration” to get

$$G\frac{mM}{r^2} = \frac{mv^2}{r} \quad \Rightarrow \quad v = \sqrt{\frac{GM}{r}}$$

for the “orbital velocity”. Note that the period of revolution is $T = 2\pi r/v$ so we have

$$T^2 = \frac{4\pi^2}{GM}r^3$$

which is called Kepler’s Law of Periods. (We’ll look at this more next week.) Note that this is a way to determine the mass of the sun from the revolution periods of the planets.

One very basic result from this is that the tangential rotation speed $v$ should be proportional to $1/\sqrt{r}$ as soon as the mass is concentrated at distances smaller than $r$. Galaxies should look like this, since there is generally a big “bulge” of bright stars near the center. However, the “galactic rotation curves” look very different! (See slides.) This is one of the strong pieces of evidence for “dark matter” in the universe.


**Practice Exercises**

1. Imagine an object with a mass $M$ and radius $R$ which is so dense that the escape velocity equals $c$, the speed of light. Write an equation for $R$ in terms of $M$ and other necessary quantities. Compare this to dimensional analysis problem we did in lecture on the first day of class.

2. Use the Milky Way galactic rotation curve to estimate the mass of the galaxy contained within a distance of 16 kpc. (Note that 1 kpc=10^3 pc=3.1 × 10^{19} m.) The mass of the sun is 2 × 10^{30} kg. How many sun-like stars do you need to get this mass? Note that the brightness of a typical spiral galaxy (like the Milky Way?) seems to be about 10^{10} times that of the sun. (Table 21-1 in Introductory Astronomy and Astrophysics by Zeilik and Gregory.)
The Milky Way Galaxy

Visible Light

Infrared Light
Before we begin, anybody want to talk about the dark matter plots some more?

Today: How Kepler’s Three Laws (ca 1605) follow from physics ala Newton (ca 1700). (Discuss a little history, including Tycho Brahe, 1546-1601.)

References: Textbook, Sec.14-7. (Just states Kepler’s Laws, does not prove them.)
See also “The special joy of teaching first year physics”, Erich Vogt, Am.J.Phys 75(2007)581
(Available to you from the web site http://scitation.aip.org/ajp/)

Today I will break my own rule, and hand out class notes before the new stuff.

Kepler’s Three Laws

1. The Law of Orbits: All planets move in elliptical orbits having the Sun at one focus. Draw a reminder of what an ellipse looks like. (Textbook Fig.14-14.) Label the semi-major axis $a$ and the distance from the focus to the center as $ea$. Discuss eccentricity $e$ a little.

2. The Law of Areas: A line joining any planet to the Sun sweeps out equal areas in equal times. Mention that this is a consequence of “conservation of angular momentum” which we discussed in class a couple of weeks ago.

3. The Law of Periods: The square of the period of any planet about the Sun is proportional to the cube of the planet’s mean distance from the Sun. We saw this was true for circular orbits, so you can guess it will hold for ellipses if we define “mean distance” appropriately.

Erich Vogt’s Proofs

The first law is the hardest to prove, so we start there. Begin with energy $E(<0)$

$$E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

and divide through by $m$ and then by the (positive) quantity $-E/m$ to find

$$\left[\frac{v^2}{2(-E/m)}\right] - \frac{GM/(-E/m)}{r} = -1$$

We will show that Eq. 6 is the equation of an ellipse. Now remember the “constant of the motion” we derived for central forces (Check Appendix H on “cross products”)

$$\ell = m(x\dot{y} - y\dot{x}) = |\vec{r} \times (m\vec{v})| = mvr \sin \phi = mvh$$

where $\phi$ is the angle between $\vec{v}$ and $\vec{r}$, so $h$ has a geometric meaning:

(Figure 1 from Vogt’s paper.) The geometry for the bound elliptical orbit of a planet at point $p$ around the Sun at the focus. The ellipse parameters ($a, b,$ and $c$) are shown as well as three alternative pairs of coordinates: $x$ and $y$, $r$ and $\theta$, $r$ and $h$, where $h$ is the perpendicular distance from the focus to the tangent at point $p$. The ellipse shown has $b = a/2$. 
Note that \( h \leq r \) everywhere along the path. Replacing \( v \) with \( h \) in Eq. 6 gives

\[
\frac{b^2}{h^2} - \frac{2a}{r} = -1
\]

(8)

where \( a = GM/2(-E/m) \) and \( b = (\ell/m)/(-2E/m)^{1/2} \). Even though it looks screwy, Eq. 8 is the same as the familiar equation for an ellipse. namely

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

(9)

Vogt demonstrates this equivalence in his paper. It’s mostly algebra with some calculus.

*This is a cool result!* It gives the semimajor (\( a \)) and semiminor (\( b \)) axes of the elliptical orbit in terms of the physical constants \( E \) and \( \ell \) for the motion.

The **second law** is simple to prove. The area \( dA \) of the skinny triangle swept out time \( dt \) is just \( h(v\,dt)/2 \) so that

\[
\frac{dA}{dt} = \frac{1}{2}hv = \frac{\ell}{2m}
\]

which is constant. Clearly, Kepler’s second law is a statement that angular momentum \( \ell \) is conserved for central forces.

The **third law** is pretty easy to prove, too. Since the area swept out per unit time \( dA/dt \) is a constant, and the area of an ellipse is \( \pi ab \), so the period must be \( T = \pi ab/(dA/dt) \). Now realize that we can write \( b = (\ell/m)(a/GM)^{1/2} \). Putting this all together, we get

\[
T = \frac{\pi a(\ell/m)(a/GM)^{1/2}}{\ell/(2m)} = \frac{2\pi a^{3/2}}{(GM)^{1/2}} \Rightarrow T^2 = \frac{4\pi^2}{GM}a^3
\]

So, it would seem that Kepler interpreted the semimajor axis \( a \) and the “mean distance” of the planet to the Sun. (It would be interesting to look up what he actually wrote and how it best translates to modern English.)

**Practice Exercises**

Just one for today. In terms of the standard equation for an ellipse, Eq. 9, the eccentricity \( e \) is given by \((a^2 - b^2)^{1/2} = ea\). (See Vogt’s paper for details.) Derive an expression for the eccentricity in terms of the constants of the motion \( E \) and \( \ell \), and the physical constants \( G \), \( M \), and \( m \). Then write \( E \) and \( \ell \) for a circular orbit of radius \( R \), and finally demonstrate that \( e = 0 \) using your expressions. (You can either plug your expressions for \( E \) and \( \ell \) into your expression for \( e \), or step back a bit and just show that they give you \( a = b \).)
First exam one week from today, in this classroom. Some relevant items:

- Homework due on Monday but not Thursday.
- See past exams, posted on the course web page.
- Lab period next week devoted to review: Bring questions!
- Don’t memorize! Bring whatever materials you’d like to the exam.

**The Simple Pendulum**

Work with the diagram in your textbook, Figure 17-10:

**Figure 17-10.** The simple pendulum. The forces acting on the pendulum are the tension $\vec{T}$ and the gravitational force $m\vec{g}$, which is resolved into its radial and tangential components. We choose the $x$ axis to be in the tangential direction and the $y$ axis to be in the radial direction at this particular time.

**First talk about the motion in terms of energy.** The potential energy is due to gravity only. (The tension $\vec{T}$ does no work on the pendulum bob, since it is always perpendicular to the direction of motion.) Let the “zero” of the potential energy be when the pendulum is hanging still. Then the potential energy due to gravity is

$$U = mg \times \text{“height”} = mg \times (L - L \cos \theta) = mgL(1 - \cos \theta)$$

The potential energy is plotted on the right. Two total energy values are indicated. Note the turning points, and emphasize that the bob will oscillate between them. This is just what you know a pendulum does. We could solve for the motion using the energy, but we won’t. Instead we’ll go back to the equation of motion, i.e. Newton’s second law, and refer back to Figure 17-10.

**To solve this with the equation of motion** first use $x$ to measure the distance along the arc, as in Fig.17-10. The only force that acts in this direction is the tangential component of gravity, namely $-mg \sin \theta$. So $F = ma$ becomes

$$-mg \sin \theta = m\ddot{x} \quad \text{or} \quad \ddot{\theta} = -\frac{g}{L} \sin \theta$$

after we cancel $m$ on both sides of the equation, and realize that $x = L\theta$. This is the differential equation we must solve in order to find $\theta(t)$, i.e. the motion of the pendulum in terms of the angle as a function of time.
However, this equation is impossible to solve exactly, and instead we resort to an approximation. If $\theta$ (which we *always* measure in radians) is small, then $\sin \theta \approx \theta$. In this case

$$\ddot{\theta} = -\omega^2 \theta$$  \hspace{1cm} (11)

where $\omega^2 \equiv g/L$. This equation is easy to solve. In general, we have

$$\theta(t) = A \cos \omega t + B \sin \omega t$$  \hspace{1cm} (12)

where $A$ and $B$ are arbitrary constants, set by the initial conditions. If the pendulum starts from rest at an angle $\theta = \theta_0$, then

$$\theta(t) = \theta_0 \cos \omega t$$  \hspace{1cm} (13)

Mention concepts of frequency $f = \omega / 2\pi$ and period $T = 1 / f = 2\pi / \omega$, and

$$T = 2\pi \sqrt{\frac{L}{g}} \times \text{corrections} = 2\pi \sqrt{\frac{L}{g}} \times \left(1 + \frac{1}{4} \sin^2 \frac{\theta_0}{2} + \cdots \right)$$

**Application to Einstein’s Equivalence Principle**

*The observation that the period of a pendulum does not depend on the mass of the bob, is a direct consequence of the equivalence of inertial and gravitational mass!* In other words, no mass appears in Eq. 10 because we canceled it on the left and right. However, if we write the gravitational force as $m_G g$ and Newton’s second law as $F = m_I a$, then we would have

$$T = 2\pi \sqrt{\frac{m_I}{m_G}} \left(\frac{L}{g}\right)$$

If anyone ever discovers that the period of a pendulum is affected by the mass of the bob (or by its composition) then it would imply that gravitational and inertial mass are not exactly the same. However, very sensitive tests have been done, and nothing has been found.

To Einstein, this was trivial. See your Textbook, Sec.14-9. Einstein did not view gravity as a “force.” Instead, it is the “shape” of space and time. That is, mass (or energy, since $E = mc^2$) makes space change in such a way that an object follows a natural path which looks as if it is being moved by a force called “gravity.” Another way of saying it, is that a projectile doesn’t follow a curved path, it follows a straight line but in a curved space. In this view, there is just one kind of mass, not two.

**Practice Exercises**

(1) Prove that Eq. 12 satisfies Eq. 11 for all values of $A$ and $B$, and that Eq. 13 satisfies the initial conditions.

(2) Write down equation for conservation of energy for a pendulum that starts from rest at $\theta = \theta_0$. Your expression should contain only $L$, $g$, $\theta$, $\dot{\theta}$, and $\theta_0$. For small $\theta$, $\cos \theta = 1 - \theta^2 / 2$. Use this to eliminate cosines from your equation, and rearrange to get an equation for $\ddot{\theta} = d\theta / dt$. Now integrate this equation to get Eq. 13.
Solution to Practice Problem 2

Problem

Write down equation for conservation of energy for a pendulum that starts from rest at \( \theta = \theta_0 \). Your expression should contain only \( L, g, \theta, \dot{\theta}, \) and \( \theta_0 \). For small \( \theta, \cos \theta = 1 - \theta^2/2 \). Use this to eliminate cosines from your equation, and rearrange to get an equation for \( \dot{\theta} = d\theta/dt \). Now integrate this equation to get Eq. 13.

Solution

The total energy \( E \), whatever it is, will not change with time and will always equal

\[
E = \frac{1}{2}mv^2 + mgL(1 - \cos \theta) = \frac{1}{2}mL^2\dot{\theta}^2 + mgL(1 - \cos \theta)
\]

where we know that the velocity at any point on the arc is \( v = \dot{x} = L\dot{\theta} \). Now since the pendulum “starts from rest at \( \theta = \theta_0 \)” we know that \( v = 0 \) when \( \theta = \theta_0 \). Therefore, the equation for conservation of energy is

\[
mgL(1 - \cos \theta_0) = \frac{1}{2}mL^2\dot{\theta}^2 + mgL(1 - \cos \theta)
\]

If the pendulum starts out at some angle \( \theta_0 \), then it will never be at an angle \( \theta \) that is greater than \( \theta_0 \). So, if we use the small angle approximation for \( \theta_0 \), then we can use it for \( \theta \), too. This turns the energy equation into

\[
\frac{1}{2}mgL\theta_0^2 = \frac{1}{2}mL^2\dot{\theta}^2 + \frac{1}{2}mgL\theta^2
\]

which is easily rearranged to give

\[
\dot{\theta} = \frac{d\theta}{dt} = -\sqrt{\frac{g}{L}} \left( \theta_0^2 - \theta^2 \right)^{1/2} = -\omega \left( \theta_0^2 - \theta^2 \right)^{1/2}
\]

which, as you know by now, can also be rearranged to give

\[
\frac{d\theta}{(\theta_0^2 - \theta^2)^{1/2}} = -\omega dt
\]

(Why did we choose to take the “-” sign when we formed the square root? Well, we are starting out at \( \theta = \theta_0 \), and the angle will decrease from there, so we expect that \( \dot{\theta} < 0 \).)

Now we want to integrate this last equation. What are the limits of integration? Well, again, we have \( \theta = \theta_0 \) when \( t = 0 \). We also have some value \( \theta \) at some time \( t \), that is, \( \theta(t) \). So, we can write

\[
\int_{\theta_0}^{\theta} \frac{d\theta'}{\left( \theta_0^2 - \theta^2 \right)^{1/2}} = -\int_{t_0}^{t} \omega dt'
\]

where we put primes on the “dummy variables” of integration. (What’s in a name?) The integral on the right is just \(-\omega t\). The integral on the left is trickier. We can make a change of variables, though, namely

\[
\theta' = \theta_0 \sin(u) \\
d\theta' = \theta_0 \cos(u) du
\]
which lets us carry out the integral on the left very simply, namely

\[ \int_0^\theta \frac{d\theta'}{\left(\theta_0^2 - \theta'^2\right)^{1/2}} = \int_{\pi/2}^{\sin^{-1}(\theta/\theta_0)} \frac{\theta_0 \cos(u) du}{\theta_0 \cos(u)} = \int_{\pi/2}^{\sin^{-1}(\theta/\theta_0)} du = \sin^{-1}(\theta/\theta_0) - \frac{\pi}{2} \]

Finally, then, we put the left and right integrals equal to each other and get

\[
\begin{align*}
\sin^{-1}(\theta/\theta_0) - \frac{\pi}{2} &= -\omega t \\
\sin^{-1}(\theta/\theta_0) &= \frac{\pi}{2} - \omega t \\
\theta/\theta_0 &= \sin\left(\frac{\pi}{2} - \omega t\right) = \cos(\omega t) \\
\text{or, finally,} \quad \theta &= \theta_0 \cos(\omega t)
\end{align*}
\]

which is indeed Eq. 13.

There are a lot of little tricks that we used to come up with this solution. (Perhaps a simpler way to approach it would be for me to ask you for the period \(T\) and you might realize that this is just four times the time for the pendulum to swing from \(\theta = \theta_0\) to \(\theta = 0\). Right?) I would not have expected you to work this all the way through, but it was worth it for you to try. You will see more and more of these sorts of mathematical manipulations as you move upward in your physics education.
First exam on Thursday. See comments from last week. (Notes posted on web site.)

Today: “Taylor Series” and their applications, including \( e^{ix} = \cos x + i \sin x \), and then on to simple harmonic motion and the solution in terms of “imaginary exponentials.”

**Math Topic: Taylor Series**

This is a way to approximate a function \( y = f(x) \) by an expansion about a point \( x_0 \) in terms of an infinite polynomial in \( (x - x_0) \) and the derivatives \( dy/dx = f'(x) \), \( d^2y/dx^2 = f''(x) \), and so forth, evaluated at \( x_0 \), i.e. \( f(x_0), f'(x_0), f''(x_0) \), etc. The formula is

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \cdots
\]

(Check that everyone knows what the factorial means, derivative notation.)

Graphic depiction of a Taylor expansion.

The function is expanded about the point \( x_0 = 1 \). Shown are the “zeroth order” (i.e. a constant) approximation, the first order, second order, and the full function. The zeroth order is just the value of the function at \( x_0 = 1 \), and the first order is the tangent line at that point. The second order approximation comes reasonably close to the full function in the neighborhood of \( x = x_0 \).

Important examples (expanded around \( x_0 = 0 \) unless otherwise stated):

\[
(1 + x)^\alpha = 1 + \alpha x + \frac{1}{2!}\alpha(\alpha - 1)x^2 + \cdots
\]

\[
e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots
\]

\[
\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots
\]

\[
\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots
\]

Obviously, if \( |x| \ll 1 \), i.e. “\( x \) is small”, then only a few terms (maybe just one) is needed in order to get an accurate approximation.

**Math Topic: Imaginary and Complex Numbers**

All you need to know for now is that there is a number \( i \) such that \( i^2 = -1 \). Here’s some more information, though, that we’ll use at some later time.

Numbers formed by a “real” number times \( i \), like \( 2i \), \( -i\sqrt{3} \), or \( xi \) where \( x \) is a “real” number, are called “imaginary” numbers. Don’t let the words fool you, though. All of these are perfectly valid numbers. The imaginary numbers are a group of numbers outside integers, rational, and irrational (real) numbers.

Numbers like \( 3 + 2i \) and \( z = x + iy \) are called “complex” numbers. They have a “real part” and an “imaginary part”. We write, for \( z = x + iy \) where \( x \) and \( y \) are real, \( x = \text{Re}(z) \) and \( y = \text{Im}(z) \). The “modulus” of \( z \) is \( |z| = \sqrt{x^2 + y^2} \).
Now, here’s a neat thing:

\[
e^{ix} = 1 + ix + \frac{1}{2!}(ix)^2 + \frac{1}{3!}(ix)^3 + \frac{1}{4!}(ix)^4 + \frac{1}{5!}(ix)^5 + \cdots
\]

\[
= \left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right] + i\left[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right]
\]

\[
= \cos(x) + i \sin(x)
\]

This is called Euler’s Formula. It is very useful in physics and engineering! Watch…

**Simple Harmonic Motion**

Probably the most important “equation of motion” in all of physics is that given by the force \( F = -kx \). This is called Hooke’s Law and describes the force from a simple spring.

The potential energy for a spring is \( U(x) = kx^2/2 \), since \( F = -dU/dx \). The potential energy has a minimum at \( x = 0 \), so an object with nonzero total energy will oscillate about the minimum. Note also that any potential energy that has a minimum at some \( x = x_0 \) can be written as the Taylor series

\[
U(x) = U(x_0) + \frac{dU}{dx}\bigg|_{x=x_0} (x - x_0) + \frac{1}{2!} \frac{d^2U}{dx^2}\bigg|_{x=x_0} (x - x_0)^2 + \cdots
\]

\[
\approx U(x_0) + \frac{1}{2} k(x - x_0)^2
\]

where \( k = U''(x_0) \), since the first derivative is zero for a minimum. That is, it looks like a spring force, if we don’t get very far away from the minimum. (The term \( U(x_0) \) doesn’t matter since we can always add a constant to \( U \) and the force doesn’t change. Also, we can just translate to \( x' = x - x_0 \) if we want to do without the ugly \( x_0 \) everywhere.)

Now, here’s the neat part. The equation of motion is just

\[-kx = m\ddot{x} \quad \Rightarrow \quad \ddot{x} = -\frac{k}{m} x\]

Try a solution of the form \( x(t) = Ae^{i\omega t} \). ("Ansatz"). Obviously \( \dot{x}(t) = i\omega e^{i\omega t} = i\omega x(t) \) and \( \ddot{x}(t) = -\omega^2 e^{i\omega t} = -\omega^2 x(t) \). In other words, our ansatz works if \( \omega = \pm \sqrt{k/m} \). We write

\[x(t) = Ae^{i\omega t} + Be^{-i\omega t}\]  

(14)

where \( A \) and \( B \) are (typically) determined by the initial conditions \( x(0) \) and \( \dot{x}(0) \). You will see this approach over and over again in your studies of physics.

**Practice Exercises**

(1) We’ve seen that the potential energy for a pendulum is \( U(\theta) = mgL(1 - \cos \theta) \). Use a Taylor expansion to show that if \( \theta << 1 \), then \( U(\theta) \) looks like the potential energy for simple harmonic motion. What is the effective value of \( k \)? What is the period \( T \)?

(2) Beginning with Eq. 14, show that a simple harmonic oscillator that starts from rest at position \( x(0) = x_0 \) moves like \( x(t) = x_0 \cos(\omega t) \).
Hand back first exam. Average over 75, so no curve. Under 60, please make an appointment to see me. Mention “line integral” problem; something similar will be on the final.

Perhaps spend a few minutes on questions regarding the exam. (Can also answer questions individually after the break.)

Today we will just scratch the surface of the subject called “oscillations.” You’ll see more of this in our course, then next semester, and then much more in your advanced courses.

**Review: Simple Harmonic Motion; Also “Phase”**

Review how we got to Eq. 14. Start with \( F = -kx \) which leads to \( \ddot{x} = -(k/m)x \). Solve this with our “ansatz” \( x(t) = e^{i\omega t} \). Plug this in to the (differential) equation of motion to find \( \omega = \pm \omega_0 \) where \( \omega_0 \equiv \sqrt{k/m} \). Warning: Slight notation changes. So, we write

\[
x(t) = x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t)
\]

Write this in a different (better?) way. Put \( B = (A/2)e^{i\phi} \) and \( C = (A/2)e^{-i\phi} \). Then

\[
x(t) = \frac{A}{2} e^{i(\omega_0 t + \phi)} + \frac{A}{2} e^{-i(\omega_0 t + \phi)} = A \cos(\omega_0 t + \phi)
\]

We call \( A \) the amplitude (“max value of \( x(t) \)” ) and \( \phi \) the phase of the oscillation.

**Drag Forces; Damped Harmonic Motion**

Not all forces are “conservative”, although you will come to appreciate that “energy” is always conserved. (We just need to be careful to include all kinds of energy.) An important example of a non-conservative force is “drag”. This is the force on an object passing through some fluid like water or air. We write

\[
F_{\text{Drag}} = -bv = -b\dot{x}
\]

where \( b \) is a (positive) constant that depends on the shape of the object and the particular fluid. Note that drag is always acting against the direction of motion, so the work it does will never be zero around a closed loop. In fact, if \( \sum F = F_{\text{Cons}} + F_{\text{Drag}} = -dU/dx - b\dot{x} \), then

\[
\frac{dE}{dt} = \frac{d}{dt} \left[ \frac{1}{2} m\dot{x}^2 + U(x) \right] = \dot{x} \left[ m\ddot{x} + \frac{dU}{dx} \right] = \dot{x} \left[ \sum F - F_{\text{Cons}} \right] = -b\dot{x}^2
\]

and the total mechanical energy always decreases. What a drag for designers.

Now let’s apply drag to our oscillator, which is no longer “simple”, but still “harmonic.” The equation of motion is

\[
\sum F = ma \Rightarrow -kx - b\dot{x} = m\ddot{x}
\]
which we rewrite as
\[ \ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = 0 \] (17)
where (again) \( \omega_0^2 \equiv k/m \) and also \( \gamma \equiv b/2m \). This looks tough to solve, but we can put our ansatz \( x(t) = e^{i \omega t} \) in and see what happens. We find from Eq. 17 that
\[ \omega^2 - 2i\gamma\omega - \omega_0^2 = 0 \]
which is easy to solve for \( \omega \). Using the quadratic equation, we get
\[ \omega = \frac{2i\gamma \pm \sqrt{-4\gamma^2 + 4\omega_0^2}}{2} = i\gamma \pm \sqrt{\omega_0^2 - \gamma^2} = i\frac{b}{2m} \pm \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \] (18)

If the oscillator is not too strongly damped, that is \( \gamma < \omega_0 \), then the solution looks like
\[ x(t) = e^{-\gamma t} \left[ Be^{i\omega_b t} + Ce^{-i\omega_b t} \right] \]
where \( \omega_b \equiv \sqrt{\omega_0^2 - \gamma^2} \) is the “frequency” of the oscillator. (Don’t forget about the 2\( \pi \)!

The amplitude of this oscillator decreases with time. (Of course it does. Its total mechanical energy is decreasing.) We call this a “damped harmonic oscillator.” The “damping time constant” is \( \tau \equiv 1/\gamma \), and is the characteristic time scale with which the amplitude decreases.

A useful measure of the damping rate is \( Q \) (for “quality factor”) defined as the energy of the oscillator at any given time, divided by the amount of energy lost over one oscillation cycle, i.e. \( Q \equiv E/\Delta E \). Your third practice problem lets you explore this quantity a bit more.

There are no oscillations if \( \gamma > \omega_0 \) (“over damping”) or if \( \gamma = \omega_0 \) (“critical damping”). For each of these cases, \( x(t) \) is made up of purely real exponential functions. (Why?) We will leave the study of these cases to an advanced course in mechanical oscillations.

**Forced Oscillation and Resonance**

This is a fascinating subject, worthy of several classes, but we won’t be able to do much with it in our course. Imagine a “driving force” \( F_{\text{Drive}} = F_0 e^{i\omega t} \). The equation of motion is
\[ \sum F = ma \Rightarrow -kx - b\dot{x} + F_0 e^{i\omega t} = m\ddot{x} \]
or \[ \ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = \left(\frac{F_0}{m}\right)e^{i\omega t} \] (19)
Try the ansatz \( x(t) = Ae^{i\omega t} \) and see how the amplitude \( A \) depends on \( \omega \), the driving frequency. (Remember that \( A \) can be complex, so consider both the “magnitude” and “phase.”) What happens when \( \omega = \omega_b \)? (This is a familiar phenomenon called “resonance.”)

**Practice Exercises**

(1) For simple harmonic motion, determine the amplitude \( A \) and phase \( \phi \) in terms of the initial position \( x_0 \) and initial velocity \( v_0 \). Sketch the two cases (i) \( v_0 = 0 \) and (ii) \( x_0 = 0 \), and indicate the phase on your sketch. Does the answer agree with your formula?

(2) What are the dimensions of \( b \) in the drag force definition, Eq. 16? How can you combine \( b \), \( m \), and \( g \) to get something with the dimensions of velocity? What do you think this velocity is physically? Prove it by solving the equation of motion for an object in free fall but subject to drag, i.e. \( \sum F = -mg - b\dot{y} \). (Why are the signs the way they are?) This is not a simple differential equation to solve, though, so ask for help if you need it.

(3) Find an expression for \( Q \) in terms of \( \gamma \) and \( \omega_0 \), that is, in terms of \( b \), \( m \), and \( k \).
New stuff today. Momentum, then applied to “systems of particles,” then on to rigid bodies. Will introduce “center of mass” (aka “center of momentum”) in terms of “three dimensional integrals.” But don’t let the fancy math language fool you!

**Momentum, Impulse, and Conservation of Momentum**

Newton actually wrote the Second Law in terms of momentum \( \vec{p} = m\vec{v} \) as

\[
\sum \vec{F} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}
\]

where we can take the mass \( m \) out of the derivative if it is a constant.

Now imagine an object colliding with a brick wall. The collision will change the momentum of the object. The force of the collision varies over the collision time. See the figure on the right. (Figure 6-7 from the textbook.) We can use Newton’s second law to write an expression involving the integral of the force and the change in momentum:

\[
\Delta \vec{p} = \int_{t_1}^{t_2} \vec{F}(t) dt \equiv \vec{J}
\]

We call \( \vec{J} \) the “impulse.” The “average force” \( \vec{F}_{avg} \) is \( \vec{J} \) divided by \( \Delta t = t_2 - t_1 \).

Now imagine two objects colliding with each other. Each exerts an “equal and opposite” momentum on each other. That is \( \vec{J}_1 = -\vec{J}_2 \), or

\[
\Delta \vec{p}_1 + \Delta \vec{p}_2 = \Delta (\vec{p}_1 + \vec{p}_2) = 0
\]

In other words, the total momentum \( \vec{P} \equiv \vec{p}_1 + \vec{p}_2 \) is conserved in collisions. Note that this doesn’t work if there is some “external” force acting on the masses.

Spend a little time talking about momentum conservation as invariance under space translation, as opposed to energy conservation as invariance under time translation.

**Many Particle Systems and Rigid Bodies**

Look at two particles again, but not just collisions. Allow some external force \( \vec{F}_{ext} \) to act as well as internal forces. Then

\[
\frac{d\vec{P}}{dt} = \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = m_1\vec{a}_1 + m_2\vec{a}_2 = \sum \vec{F}_{ext}
\]

The last equality follows because the internal forces cancel in the sum. It is convenient to write \( \vec{P} = M\vec{v}_{cm} \) where \( M = m_1 + m_2 \) and \( \vec{v}_{cm} = d\vec{r}_{cm}/dt \) with

\[
\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \frac{1}{M} (m_1\vec{r}_1 + m_2\vec{r}_2)
\]
in which case Newton’s Second Law reads \( \sum \vec{F}_{\text{ext}} = M \ddot{\vec{r}}_{\text{cm}} = M \vec{a}_{\text{cm}} \). We call \( \vec{r}_{\text{cm}} \) the center of mass, although a better name is probably center of momentum.

This all carries through if there are more than two particles in the “system.” We write

\[
\vec{r}_{\text{cm}} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \cdots + m_N \vec{r}_N}{m_1 + m_2 + \cdots + m_N} = \frac{1}{M} \sum_{n=1}^{N} m_n \vec{r}_n \quad (21)
\]

Don’t forget that this is really two (or three) equations for \( x_{\text{cm}}, y_{\text{cm}}, \) and (if three) \( z_{\text{cm}} \). (See textbook, Eq.7-12.) See figures 7-8 and 7-9, which show how this all works out:

Calculating the Center of Mass of Solid Objects

If we are working on a solid object, we could sum over all the atoms that make it up, but that would be nuts. Instead, we use calculus, and assume that the object is continuous. Then the sum becomes an integral. That is

\[
M = \int dm \quad \text{and} \quad \vec{r}_{\text{cm}} = \frac{1}{M} \int dm \ \vec{r}
\]

We usually write the infinitesimal mass \( dm \) as a density (usually called \( \rho \) or \( \sigma \)) times a volume element \( dV \) (or some other measure), i.e. \( dm = \rho(\vec{r})dV \). This allows the density to vary as a function of position inside the solid object. The rest is in the details.

Here is a simple example. What is the volume of a right circular cone of height \( h \) and base radius \( R \)? Let \( z \) measure the height, and slice the cone up into thin disks, each of thickness \( dz \). The radius of each disk is \( R(h - z)/h = R(1 - z/h) \), so the volume of each disk is \( \pi R^2 (1 - z/h)^2 dz \). Therefore, the volume of the cone is

\[
\int dV = \int_0^h \pi R^2 (1 - z/h)^2 dz = \pi R^2 \int_0^h \left( 1 - \frac{2z}{h} + \frac{z^2}{h^2} \right) dz = \pi R^2 \frac{h}{3}
\]

or one-third of the base area times the height. Of course this is volume, not mass, but that’s basically the same thing for constant density. If the density is not a constant, you’d need to include the functional form of \( \rho(\vec{r}) \) and that would be a harder problem.

Practice Exercises

1. An “elastic” collision is one in which kinetic energy is conserved. Let two objects, one with mass \( m \) and the other with mass \( 2m \) collide elastically in one dimension. They are initially heading towards each other with speed \( v \). Find their speeds and directions after the collision, in terms of \( m \) and \( v \).

2. Find the position of the center of mass of a long thin rod of mass \( M \) and length \( L \). The density of the material making up the rod changes linearly from zero at one end. Start by realizing that the mass of a short segment of length \( dx \) is \( dm = \sigma(x)dx \) where \( \sigma(x) \) is the linear mass density. Write a formula for \( \sigma(x) \), and determine any parameters by fixing the mass of the rod to be \( M \). Then, find the position of the center of mass.
Today: Describing the motion of solid objects, also known as “rigid bodies.” Note: It is convenient to describe rotational motion about the center of mass. Then, let the object move according to external forces by following the CM, and deal with rotations about the CM separately. However, we don’t have to do it this way!

**Kinematics: Describing the Motion of Solid Objects**

Specify an “axis of rotation.” Then motion of all parts of the body are described by some angle of rotation $\phi = s/r$ about that axis, even though $s$ and $r$ will be different for different pieces of the object. See Fig. 8-3.

So, $\phi$ takes the place of displacement $x$ for one dimensional “translational” motion. Change in $\phi$ is $\Delta \phi$ and “angular velocity” is $\omega \equiv d\phi/dt = \dot{\phi}$ and “angular acceleration” is $\alpha \equiv d\omega/dt = \ddot{\omega}$.

Everything else follows. For example, rotation under constant acceleration is (Eq. 8-7)

$$\phi(t) = \frac{1}{2} \alpha t^2 + \omega_0 t + \phi_0$$

Be careful of units! The “correct” units are radians, which are truly dimensionless. Sometimes people use degrees ($180^\circ = \pi$ radians) or revolutions ($2\pi$ radians), so watch out.

Is this the “same” $\omega$ we used in simple harmonic motion? Yes! If the angular velocity is constant, then the time $T$ it takes for one complete revolution is given by $\omega T = 2\pi$.

**Vector Representations of Rotational Kinematics**

We can write $\vec{\omega}$ or $\vec{\alpha}$, but we can’t write $\vec{\phi}$! This is because (finite) rotations do not commute. (Do the trick with the textbook.) However, tiny rotations do commute, so $d\vec{\phi}$ is a vector, and so is $\vec{\omega} = d\vec{\phi}/dt$. Since $\vec{\omega}$ is a vector, $\vec{\alpha} = d\vec{\omega}/dt$ is, too.

So what direction is $\vec{\omega}$? It has to be along the axis, since that’s the only direction that isn’t changing. By convention, we use the “right hand rule” to choose the positive direction.

We won’t stress it, but it is good to see the relationships between linear and angular variables, especially in vector form. See Figs 8-10 and 8-11 in your textbook:

For example, $\vec{v} = \vec{\omega} \times \vec{R}$, where $\vec{v}$ must be specified at some point within the object, located by $\vec{R}$. Also, $\vec{a} = \vec{\alpha} \times \vec{R} + \vec{\omega} \times \vec{v}$. Do you see what happens if $\vec{\omega}$ is constant? (See practice.)
Dynamics: Torque and Rotational Inertia

Forces cause translational motion, but torques cause rotational motion. The torque \( \tau \) from a force \( \vec{F} \) applied at a point specified by \( \vec{r} \) is \( \tau = rF \sin \theta = r(F \perp) = (r \perp)F \) where \( \sin \theta \) is the angle between \( \vec{r} \) and \( \vec{F} \). We call \( r \perp = r \sin \theta \) the “moment arm” of the force. Clearly \( \theta = 0 \) means no torque, no matter what the force.

In vector notation \( \vec{\tau} = \vec{r} \times \vec{F} \).

Imagine a bunch of masses “hanging” from a string. The torque due to gravity is

\[
\sum_n \vec{r}_n \times \vec{F}_n = \sum_n \vec{r}_n \times (-m_n \vec{g}) = \left( \sum_n m_n \vec{r}_n \right) \times (-\vec{g}) = \vec{r}_{cm} \times (-\vec{g})
\]

If we hang from the center of mass, \( \vec{r}_{cm} = 0 \). So, there is no torque and the object is balanced. Sometimes we call the center of mass the “center of gravity.”

So what is the motion resulting from a torque? Consider one single particle, inside the solid object, located at \( \vec{r} \) and acted on by \( \vec{F} \). It can only move in a circle of radius \( r \) and the force in that direction is \( F \sin \theta \). If the path length it travels is called \( s \), then Newton’s Second Law says \( F \sin \theta = m \ddot{s} = mr \ddot{\phi} \). Multiply both sides by \( r \), and we have \( \tau = (mr^2) \alpha \). The quantity \( mr^2 \) has replaced mass when we went from translational to rotational motion.

Since \( \alpha \) is the same for all parts of the object, and since \( \tau \) is either from one force at one point, or due to a sum of such things, Newton’s Second Law for a solid object becomes

\[
\sum \vec{\tau} = I \vec{\alpha}
\]

where \( I = \sum_n m_n r_n^2 \) (22)

is called the rotational inertia or moment of inertia. Notice that \( I \) is a property of the solid object and the specified axis. (Do you get the feeling that \( I \) is not really a “scalar” but not a vector either? If so, you are right. It is actually something called a “tensor.”)

You can prove something called the Parallel Axis Theorem which states that the rotational inertia of any body about an arbitrary axis equals the rotational inertia about a parallel axis through the center of mass plus the total mass times the squared distance between the two axes. Mathematically, we write \( I = I_{cm} + M h^2 \).

**Rotational Inertia of Solid Objects**

Of course, for solid objects, we don’t do the sum over all the masses in Eq. 22 to get the rotational inertia. Instead, we do the integral \( I = \int r^2 dm \). This is just like calculating the center of mass, that is \( \int r dm \), but with another factor of \( r \). See Fig. 9-15.

**Practice Exercises**

1. Follow Figure 8-11 and show that for an object spinning at some constant angular speed \( \omega \), then the acceleration \( \vec{a} \) at some point has the correct direction and magnitude for the centripetal acceleration.

2. Find the rotational inertia of a long thin homogeneous rod of mass \( M \) and length \( L \), about an axis through one end and perpendicular to its length. Now use the parallel axis theorem to get the rotational inertia for an axis through the center.
Hand in homework to Eli.

Work more on yesterday’s lab! Let me know if you want time to make measurements. I will key on this lab when grading your lab books, since this is a comprehensive experiment.

**Angular Momentum of a Single Particle**

Recall Eq. 3, i.e. $\ell \equiv m(x\dot{y} - y\dot{x})$ turns out to be a “constant of the motion” if particle acted on by a central force. Mention concept of “rotational symmetry” and central forces.

Now we know that $\ell$ is the z-component of the vector (i.e. “cross”) product of $\vec{r} = x\hat{i} + y\hat{j}$ and $\vec{p} = m\vec{v} = m(\dot{x}\hat{i} + \dot{y}\hat{j})$, that is $\ell = |\vec{r} \times \vec{p}|$. (See Appendix H.) Therefore $\ell = mvr \sin \phi$ where $\phi$ is the angle between $\vec{r}$ and $\vec{p}$. (Draw figure.) When we did Vogt’s derivation of Kepler’s Laws, we used $\ell = mvh$ where $h = r \sin \phi$. (See Eq. 7.) If instead, we wrote $r_\perp$ instead of $h$, then $\ell = mvr_\perp = pr_\perp$. Thus, “angular momentum” is the “moment of momentum” just as “torque” is the “moment of force.”

Next we concern ourselves with the vector $\vec{I} = \vec{r} \times \vec{p}$ for a bit of mass $m$ that is part of a rigid body rotating about a fixed axis. See Figure 10-6 in your textbook:

The rotation is at a constant angular velocity $\vec{\omega}$. However, the angular momentum vector $\vec{I}$ is not constant. It has a constant component $\ell_z$ and a perpendicular component $\vec{l}_\perp$ which rotates. (The force $\vec{F}$ from the tension in the horizontal rod supplies the torque which changes $\vec{I}$.) For the $z$-component, we can write

$$\ell_z = \ell \sin \theta = mvr \sin \theta = mvr' = (mr')^2(v/r') = I_z\omega$$

where $I_z = mr'^2$ is the rotational inertia $I$ for a rotation about the $z$-axis, and $\omega$ is the (magnitude of the) angular velocity (for the whole rigid body).

This result is kind of nice. It is sort of like $p = mv$ but using angular variables instead of translational ones. Of course, we would like to write something like $\vec{I} = I\vec{\omega}$ but this would not be true, at least not for the single mass $m$.

Consider however a symmetric rigid body. (See Figure 10-7 in your book.) For every mass $m$ there is one symmetrically placed on the opposite side of the axis. In this case, everything except the $\ell_z$ components will cancel when summing up the masses (or integrating) over the whole solid object. Therefore the total angular momentum $\vec{L} = I\vec{\omega}$ for symmetric rigid bodies. The symmetry must be around the axis of rotation.
Newton’s Second Law for Angular Variables

What changes the angular momentum $\vec{\ell}$ for a single particle? Let’s see:

$$\frac{d\vec{\ell}}{dt} = \frac{d}{dt} (\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times (m\vec{v}) + \vec{r} \times \vec{F} = \vec{r} \times \vec{F} = \vec{\tau}$$

where we used Newton’s Second Law, i.e. $\vec{F} = d\vec{p}/dt$. Well, this is nice. Newton’s Second Law for a single particle looks the same in angular variables as it does for translational variables. (This is what I meant when I said “the tension in the horizontal rod supplies the torque which changes $\vec{\ell}$” in the discussion above.)

For a system of particles, we can do the same thing for the total angular momentum

$$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \cdots + \vec{\ell}_n = \sum_n \vec{\ell}_n \quad \Rightarrow \quad \frac{d\vec{L}}{dt} = \sum_n \vec{\tau}_n$$

However, in the sum over torques, all of the internal torques will cancel, since they will all be paired up at the same point but with equal and opposite forces. So, this last equation can actually be written as the sum over external torques alone, that is

$$\sum \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt} \quad (23)$$

This is analogous to Newton’s Second Law for a system of particles, in terms of the center of mass and external forces. There is actually a big surprise, though, from this equation.

Consider the behavior of the bicycle wheel in Figure 10-5. One end of the axle is on a pedestal. If you hold the other end, nothing odd happens. But if you let go of the end, the wheel doesn’t fall. It precesses around the pedestal!

This is just what you expect (!?) from Eq. 23. For a spinning wheel, $\vec{L}$ is directed along the axle. When you let go, the only external torque around the pedestal pivot point is from gravity. The direction of the torque ($= \vec{r} \times (M\vec{g})$) is sideways. This is the direction in which the vector $\vec{L}$ changes. This is the same principle as a spinning top. (See Section 10-5.)

Of course, the classic application of Eq. 23 is the conservation of angular momentum. If there are no external torques, then $d\vec{L}/dt = 0$ and $\vec{L}$ is a constant. Skaters speed up their spinning if they pull in their arms, for example. See also Figures 10-12, 10-13, and 10-14 for more examples.

Practice Exercises

(1) Consider a simple pendulum, with a mass $m$ hanging from a string of length $L$. Write expressions for the total torque $\sum \tau$ on the mass and its angular momentum $\ell$ in terms of the pendulum angle $\theta$. Show that $\sum \tau = d\ell/dt$ yields the same equation of motion for the pendulum as in Eq. 10, i.e. $\ddot{\theta} = -(g/L) \sin \theta$. (See Sample Problem 10-2 in your book.)

(2) Spinning neutron stars are sometimes observed as “pulsars”, intense radio astrophysical objects with periods of fractions of a second. The sun is (almost) a sphere, which rotates once every 25 days. A neutron star is made up of tightly packed neutrons. If the Sun collapsed into a neutron star, what would be its rotation period? (A neutron occupies a cube with side length about $2 \times 10^{-15}m$. See Appendices B and C for whatever else you need.)
New stuff today. Motion of “fluids.” Some gases, mainly incompressible liquids. Thursday we will look more closely at the mathematics of “vector fields.”

*Note: This is an introduction to “continuum mechanics”, a.k.a. “classical field theory.”*

**How to Look at a Fluid**

Fluids are continuous, but not rigid. Instead of “force” and “mass” we talk about “pressure” and “density” to formulate kinematics and dynamics. We will refer to “regions” of a fluid, that are enclosed by a “surface” (which doesn’t have to be “real.”)

Force on a small area element of a surface is $\Delta \vec{F}$. Size of area is $\Delta A$. Define vector area element $\Delta \vec{A}$ by giving it a direction that points outward from the enclosed volume. Then

$$\Delta \vec{F} = p \Delta \vec{A}$$

where $p$ is the pressure. The SI unit is a “pascal” (Pa). Atmospheric pressure is just about $1.0 \times 10^5$ Pa, at the ocean bottom $\sim 1.0 \times 10^8$ Pa. See Table 15-1 for more examples.

A small volume element $\Delta V$ will have a mass $\Delta m$. The density $\rho$ is just the mass per unit volume, so $\rho = \Delta V/\Delta m$. If $\rho$ is a constant throughout a fluid of total volume $V$ and mass $m$, then $\rho = m/V$.

Volume changes in response to pressure. If something of volume $V$ changes by an amount $\Delta V$ in response to a pressure change $\Delta p$, then the value of

$$B = -\frac{\Delta p}{\Delta V/V} > 0$$

is pretty much the same for all volumes of any given material. It is called the “Bulk Modulus.” For liquids, $B \sim 10^9$ N/m$^2$, a number so large that we generally speak of “incompressible” liquids. For gases, $B \sim 10^5$ N/m$^2$, much smaller.

Consider a static fluid at rest but under gravity. The weight of the fluid “on top” increases the pressure underneath. Consider the forces on a horizontal slug of fluid (Fig.15-2):

Up is positive. Mass is $dm = \rho dy$, so gravity is $-\rho g Ady$. Pressure up on bottom face is $+pA$. Pressure down on top face is $-(p+dp)A$. The sum of y-forces is zero, so

$$+pA - (p + dp)A - \rho g Ady = 0$$

$$\frac{dp}{dy} = -\rho g \quad (24)$$

For a constant density fluid (like water) this means that $\Delta p = -\rho g \Delta y$, or $p = p_0 + \rho gh$ where $h$ is the “depth” below the surface of the water pool. For an “ideal gas” (remember from chemistry, or wait a few weeks) $\rho$ is proportional to $p$, i.e. $\rho = kp$. Then $dp/dy = -kgp$ so

$$p(y) = p_0 e^{-ky}$$

where $p_0$ is the pressure at “sea level.” Also $\rho_0 = kp_0$ is the density at sea level, so $p(y) = p_0 e^{-y/a}$ where $a = 1/kg = p_0/\rho_0 g \approx 9$ km for the Earth’s atmosphere.
Pascal’s Principle and Archimede’s Principle

Pascal noticed that pressure transmitted itself equally through the volume of a fluid. This is “obvious” for an incompressible fluid, but true in general. Typical application is the “hydraulic lever” pictures to the right. (Figure 15-8.) See Figure 15-9 for an example of a realistic car jack, with a fluid reservoir and alternating values.

Archimedes (“Eureka!”) realized that a body will sink until it displaces enough fluid to displace its own weight. This is easy to see. Imagine a blob of water inside the tank, surrounded by identical water. (Figure 15-10). It doesn’t move, so forces all balance. Now replace the water blob with another material, so the blob now has mass \( m \). It sinks if the density is greater, or floats if the density is less.

The Continuity Equation and Bernoulli’s Equation

Some mass \( \delta m \) flows through a “screen” of area \( A \) in a time \( \delta t \). The volume of water swept through is just \( A \times v \delta t \) where \( v \) is the speed flowing perpendicular to the plane of \( A \). Obviously \( \delta m = \rho Av \delta t \). We call \( \delta m/\delta t = \rho Av \) the “mass flux.”

So what? The point is that mass is conserved, so the mass flux has to be the same at any point in a flow. This means that \( \rho Av \), or just \( Av \) if the fluid is incompressible, must be the same everywhere. This is called the equation of continuity and is very useful practically.

The heart of fluid dynamics is something called Bernoulli’s Equation. It combines the concepts of fluid flow with conservation of energy. See Figure 16-6 in your textbook:

A blob of water with mass \( \delta m \) is pushed through the pipe in some time \( \delta t \). The kinetic energy change is \( \Delta K = \frac{1}{2} \delta m (v_2^2 - v_1^2) \). The work done by gravity is \( -\delta mg (y_2 - y_1) \) and the work done by the difference in pressure is \( +p_1 A_1 \delta x_1 - p_2 A_2 \delta x_2 = (p_1 - p_2) \delta m / \rho \).

So, the work-energy theorem \( W = \Delta K \) implies that

\[
p + \frac{1}{2} \rho v^2 + \rho gy = \text{constant} \quad (25)
\]

This is Bernoulli’s equation.

Practice Exercises

(1) As water falls from a faucet, it “necks down”. Use the continuity equation and conservation of energy to explain why. From the ratio \( \alpha = A_1 / A_2 \) in the figure on the right, derive an expression for the speed \( v_1 \) in terms of \( \alpha \) and \( h \).

(2) Consider a 1m\( \times \)1m window in a tall building. A 30 mph wind blows past the window, tangential to its surface. Use Eq. 25 to calculate the outward force pushing on the window pane. You can assume atmospheric pressure on either side of the window when no wind is blowing. (You may want to look up the John Hancock Tower in WikiPedia.)
Today: Vector fields, applied to fluids, but with textbook figures from E&M (Chap.27).

**Fields, Vector Fields, and Flux**

A field is any (physical) function of space, and maybe time, too. Simple example is density \( \rho = \rho(x, y, z, t) = \rho(\vec{r}, t) \). A vector field is just some vector function \( \vec{v} = \vec{v}(\vec{r}, t) \). It has (potentially) a different magnitude and direction at any point in space (and time). We will use \( \vec{v} \) for a generic vector field, and also for the velocity field of a fluid. Another vector field we’ll see today is \( \vec{j} \equiv \rho \vec{v} \).

We show vector fields with “flow patterns” or “field lines,” Figs.16-13 and 16-16. (See also Figs.16-14 and 16-15.) Closer together lines, means larger the magnitude. Fig.16-13 shows a vector field \( \vec{v}(\vec{r}, t) = \text{constant} \). Fig.16-16 shows a “source.”

We speak of the “flux” \( d\phi \) of a vector field through some area \( dA \). For flow perpendicular to the area, \( d\phi = |\vec{v}|dA \). If the plane is at some angle \( \theta \), then \( d\phi = |\vec{v}|dA(\cos \theta) \). Put all this together, and write \( d\phi = \vec{v} \cdot d\vec{A} \). See Fig.27-1 below. (Yes, this is in Honors Physics III!)

![Diagram of vector fields and flux](image)

It should be obvious to you by now, that all we need to do to find the flux through some large area \( A \) is to integrate over the area. Symbolically, \( \phi = \int d\phi = \int \vec{v} \cdot d\vec{A} \), but the details will always depend on the specific area and vector field involved.

**The Divergence Theorem (aka Gauss’ Theorem)**

If the area over which we are integrating is a closed surface, then we write \( \phi = \oint \vec{v} \cdot d\vec{A} \). See Fig.27-2 for picture. At each little piece of area (which they call \( \Delta \vec{A} \) instead of \( d\vec{A} \)) there is a piece of flux. To do the integral, we add up all those pieces. Once again, the details depend on the area and the vector field.

There is an important theorem which will prove and use:

\[
\oint \vec{v} \cdot d\vec{A} = \int (\nabla \cdot \vec{v}) \, dV \quad (26)
\]
where the integral on the right is over the volume enclosed by the area, and

\[ \nabla \cdot \vec{v} = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \]  

(27)

is called the “divergence” of \( \vec{v}(x, y, z) \). We’ll explain all this as we go through the proof.

Our “proof” is simple. Imagine a very small enclosed surface, a box with faces parallel to the axis planes, with one corner at \((x, y, z)\) and the other at \((x + dx, y + dy, z + dz)\). Draw a picture. Since the box is small, we replace the integral with a sum:

\[
\oint \vec{v} \cdot d\vec{A} = -v_x(x, y, z)(dydz) + v_x(x + dx, y, z)(dydz) \\
- v_y(x, y, z)(dxdz) + v_y(x, y + dy, z)(dxdz) \\
- v_z(x, y, z)(dxdy) + v_z(x, y, z + dz)(dxdy) \\
= \left[ \frac{v_x(x + dx, y, z) - v_x(x, y, z)}{dx} \right] (dxdydz)
\]

which proves the theorem for the tiny box. If we build a big volume out of tiny boxes, then the flux in one side of a box is the negative of the flux out of the side of its neighbor, so everything but the overall surface cancels.

The Continuity Equation (for real, this time)

Now remember the mass flux \( \delta m/\delta t = \rho Av \). For a tiny area \( dA \) we can write this is \( \vec{j} \cdot d\vec{A} \) where \( \vec{j} \equiv \rho \vec{v} \), so the mass flowing through an arbitrary surface per unit time is \( \int \vec{j} \cdot d\vec{A} \).

Mass is neither created nor destroyed, so the net mass flowing through a closed surface must be zero. (This assumes there is no “source” of mass, like the vector field in Fig.16-16. Maybe it would be the end of a long skinny hose. The concept will become more important in Physics II.)

The enclosed mass is just the integral of density over volume. So,

\[
\frac{d}{dt} M_{\text{enc}} = -\oint \vec{j} \cdot d\vec{A} \quad \Rightarrow \quad \frac{d}{dt} \int \rho dV = -\int (\nabla \cdot \vec{j}) dV
\]

using the Divergence Theorem, Eq. 26. The “-” sign is because positive flux is mass flowing “out”, which decreases the mass. Bringing the time derivative inside makes it a partial derivative. Then, realizing this is true for any volume, gives us

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0
\]

(28)

This is the Continuity Equation, written in a more formal way. Early, we just wrote that \( \rho Av \) was a constant, which is just \( \oint \vec{j} \cdot d\vec{A} = 0 \) for a long, straight tube in one dimension \( x \). The more formal statement says this differently. Picture a tube with a linear taper in the \( z \) direction. For constant density, \( \partial v_x/\partial x < 0 \) and \( \partial v_y/\partial y < 0 \) so \( \partial v_z/\partial z > 0 \) to keep \( \nabla \cdot \vec{j} = 0 \). That is, the speed increases.

Practice Exercise

(See Textbook 16P13.) A fluid is called “irrotational” if the closed line integral \( \oint \vec{v} \cdot d\vec{s} \) is zero around some path. Choose a rectangular closed path, and show that \( \oint \vec{v} \cdot d\vec{s} = 0 \) for the velocity field in Fig.16-13 (above), and that \( \oint \vec{v} \cdot d\vec{s} \neq 0 \) for the velocity field on the right. That is, the field on the right is “rotational.”
Second Midterm Exam is This Thursday!

The topic we call “coupled oscillations” has far reaching implications. The formalism ends up being appropriate for many different applications, some of which bear only a passing resemblance to classical oscillation phenomena. This includes the mathematics of eigenvalues and eigenvectors, for example.

These notes describe the elementary features of coupled mechanical oscillations. We use one specific example, and describe the method of solution and the physical implications implied by that solution. More complicated examples, and their solutions, are easy to come by, for example on the web.¹

The following figure describes the problem we will solve:

![Diagram of coupled oscillators with masses and springs](http://math.fullerton.edu/mathews/n2003/SpringMassMod.html)

Two equal masses \( m \) slide horizontally on a frictionless surface. Each is attached to a fixed point by a spring of spring constant \( k \). They are “coupled” to each other by a spring with spring constant \( k_c \). The positions of the two masses, relative to their equilibrium position, are given by \( x_1 \) and \( x_2 \) respectively.

Now realize an important point. We have two masses, each described by their own position coordinate. That means we will have two equations of motion, one in terms of \( \ddot{x}_1 \) and the other in terms \( \ddot{x}_2 \). Furthermore, since the motion of one of the masses determines the extent to which the spring \( k_c \) is stretched, and therefore affects the motion of the other mass, these two equations will be “coupled” as well. We will have to develop some new mathematics in order to solve these coupled differential equations.

We get the equations of motion from “\( F = ma \)”, so let’s do that first for the mass on the left, i.e., the one whose position is specified by \( x_1 \). It is acted on by two forces, the spring \( k \) on its left and the spring \( k_c \) on its right. The force from the spring on the left is easy. It is just \( -kx_1 \).

The spring on the right is a little trickier. It will be proportional to \( (x_2 - x_1) \) since that is the extent to which the spring is stretched. (In other words, if \( x_1 = x_2 \), then the length of spring \( k_c \) is not changed from its equilibrium value.) It will also be multiplied by \( k_c \), but we need to get the sign right. Note that if \( x_2 > x_1 \), then the spring is stretched, and the force on the mass will be to the right, i.e. positive. On the other hand, if \( x_2 < x_1 \), then the spring is compressed and the force on the mass will be negative. This makes it clear that we should write the force on the mass as \( +k_c(x_2 - x_1) \).

The equation of motion for the first mass is therefore

\[
-kx_1 + k_c(x_2 - x_1) = m\ddot{x}_1
\]

¹See [http://math.fullerton.edu/mathews/n2003/SpringMassMod.html](http://math.fullerton.edu/mathews/n2003/SpringMassMod.html).
We get some extra reassurance that we got the sign right on the second force, because if $k = k_c$, then the term proportional to $x_1$ does not cancel out.

The equation of motion for the second mass is now easy to get. Once again, from the spring $k$ on the right, if $x_2$ is positive, then the spring pushes back so the force is $-kx_2$. The force from the “coupling” spring is the same magnitude as for the first mass, but in the opposite direction, so $-k_c(x_2 - x_1)$. This equation of motion is therefore

$$-kx_2 - k_c(x_2 - x_1) = m\ddot{x}_2$$

So now, we can write these two equations together, with a little bit of rearrangement:

$$\begin{align*}
(k + k_c)x_1 - k_c x_2 &= -m\ddot{x}_1 \\
-k_c x_1 + (k + k_c)x_2 &= -m\ddot{x}_2
\end{align*}$$

To use some jargon, these are coupled, linear, differential equations. To “solve” these equations is to find functions $x_1(t)$ and $x_2(t)$ which simultaneously satisfy both of them. We can do that pretty easily using the exponential form of sines and cosines, which we discussed earlier when we did oscillations with one mass and one spring. Following our noses, we write

$$x_1(t) = a_1 e^{i\omega t} \quad x_2(t) = a_2 e^{i\omega t}$$

where the task is now to see if we can find expressions for $a_1$, $a_2$, and $\omega$ which satisfy the differential equations. You’ll recall that taking the derivative twice of functions like this, brings down a factor of $i\omega$ twice, so a factor of $-\omega^2$. Therefore, plugging these functions into our coupled differential equations gives us the algebraic equations\(^2\)

$$\begin{align*}
(k + k_c)a_1 - k_c a_2 &= m\omega^2 a_1 \\
-k_c a_1 + (k + k_c)a_2 &= m\omega^2 a_2
\end{align*}$$

This is good. Algebraic equations are a lot easier to solve than differential equations. To make things even simpler, let’s divide through by $m$, do a little more rearranging, and define two new quantities $\omega_0^2 \equiv k/m$ and $\omega_c^2 \equiv k_c/m$. Our equations now become

$$\begin{align*}
(\omega_0^2 + \omega_c^2 - \omega^2)a_1 - \omega_c^2 a_2 &= 0 \\
-\omega_c^2 a_1 + (\omega_0^2 + \omega_c^2 - \omega^2)a_2 &= 0
\end{align*}$$

So, what have we accomplished? We think that some kind of conditions on $a_1$, $a_2$, and $\omega$ will solve these equations. Actually, we can see a solution right away. In mathematician’s language, these are two coupled homogeneous (i.e. $= 0$) equations in the two unknowns $a_1$ and $a_2$. The solution has to be $a_1 = a_2 = 0$. Yes, that is a solution, but it is a very boring one. All it means as that the two masses don’t ever move.

The way out of this dilemma is to turn these two equations into one equation. In other words, if the left hand side of one equation was a multiple of the other, then both equations would be saying the same thing, and we could solve for a relationship between $a_1$ and $a_2$.

\(^2\)A mathematician would call collectively call these two equations an eigenvalue equation. The reason becomes clearer when you write this using matrices.
but not $a_1$ and $a_2$ separately. Mathematically, this condition is just that the ratio of the coefficients of $a_2$ and $a_1$ for one of the equations is the same as for the other, namely\(^3\)

\[
\frac{-\omega_c^2}{\omega_0^2 + \omega_c^2 - \omega^2} = \frac{\omega_0^2 + \omega_c^2 - \omega^2}{-\omega_c^2}
\]

\[
\omega_0^2 = (\omega_0^2 + \omega_c^2 - \omega^2)^2
\]

\[
\pm \omega_c^2 = \omega_0^2 + \omega_c^2 - \omega^2
\]

\[
\omega^2 = \frac{\omega_0^2 + \omega_c^2 - \omega^2}{2}\]

In other words, these two equations are really one equation if

\[
\omega^2 = \omega_0^2 \equiv \omega_A^2
\]

or, instead, if

\[
\omega^2 = \omega_0^2 + 2\omega_c^2 \equiv \omega_B^2
\]

Borrowing some language from the mathematicians, the physicist refers to $\omega_A^2$ and $\omega_B^2$ as “eigenvalues.” We will also refer to $A$ and $B$ as “eigenmodes.”

What is the physical interpretation of the eigenmodes? To answer this, we go back to Eq. 29 or, equivalently, Eq. 30. (Remember, they are the same equation now.) If we substitute $\omega^2 = \omega_0^2$ into Eq. 29 we find that

\[
a_1^A = a_2^A
\]

where the superscript just marks the eigenmode that we’re talking about. In other words, the two masses move together in “lock step”, with the same motion. This happens when the frequency is $\omega = \pm \omega_0 = \pm \sqrt{k/m}$, and that is just what you expect. It is as if the two masses are actually one, with mass $2m$. The effective spring constant is $2k$, as you learned in your laboratory exercise. Therefore, we expect this double-mass to oscillate with $\omega^2 = (2k)/(2m) = k/m$. Good. This all makes sense.

Now consider eigemode $B$. Substituting $\omega^2 = \omega_0^2 + 2\omega_c^2$ into Eq. 29, we find that

\[
a_1^B = -a_2^B
\]

In other words, the two masses oscillate “against” each other, and with a somewhat higher frequency. In this case, it is interesting to see the difference between $\omega_c \ll \omega_0$ (in other words, a very weak coupling spring) and $\omega_c \gg \omega_0$ (strong coupling spring). For the weak coupling spring, the frequency is just the same as $\omega_0$, and that makes sense. If the two masses aren’t coupled to each other very strongly, then they act as two independent masses $m$ each with their own spring constant $k$. On the other hand, for a strong coupling between the masses, the frequency is $\omega = \omega_c \sqrt{2}$, which gets arbitrarily large. Sometimes, this “high frequency mode” can be hard to observe.

A simple experimental test of this result is to set up two identical masses with three identical springs. In other words, $\omega_c = \omega_0$. In that case $\omega_B^2 = 3\omega_A^2$ and therefore the frequency for mode $B$ should be $\sqrt{3}$ times larger than the frequency for mode $A$. We should be able to make this test as a demonstration in class.

We have been talking about general properties of the two eigenmodes. Of course, this doesn’t tell you how the system behaves given certain starting values, that is, specific initial conditions.

\(^3\)A mathematician would say that we are setting the determinant equal to zero.
For this, we need to understand that the two eigenmodes can be combined. In fact, the general motion of each of the masses really needs to be written as a sum of the two eigenmodes. Each of these comes with a positive and negative frequency, just as it did when we applied all this to the motion of a single mass and spring. (Recall that the sum and difference of a positive and negative frequency exponential, are equivalent to a cosine and sine of that frequency.) Incorporating what we’ve learned about the relative motions of modes $A$ and $B$, we can write the motions as follows:

\[
\begin{align*}
x_1(t) &= ae^{i\omega_A t} + be^{-i\omega_A t} + ce^{i\omega_B t} + de^{-i\omega_B t} \\
x_2(t) &= ae^{i\omega_A t} + be^{-i\omega_A t} - ce^{i\omega_B t} - de^{-i\omega_B t}
\end{align*}
\]

where $a$, $b$, $c$, and $d$ are constants which are determined from the initial conditions.

An obvious set of initial conditions is starting mass #1 from rest at $x = x_0$, and with mass $x_2$ from rest at its equilibrium position. This leads to the solution

\[
\begin{align*}
x_1(t) &= (x_0/2) \left[ \cos(\omega_A t) + \cos(\omega_B t) \right] \\
x_2(t) &= (x_0/2) \left[ \cos(\omega_A t) - \cos(\omega_B t) \right]
\end{align*}
\]

**Practice Exercises**

1. Show that these equations satisfy the equations of motion and the initial conditions.

2. Make a plot of $x_1(t)$ and $x_2(t)$. Choose some value for $x_0$ and $\omega_A$. Also choose a value for $\omega_B$ that represents a “weak coupling” system. Explain the motion of each of the two masses in terms of conservation of mechanical energy. It would be helpful if you rewrite Eqs. 31 and 32 using the trigonometric identities

\[
\begin{align*}
\cos(u) + \cos(v) &= 2 \cos\left(\frac{u + v}{2}\right) \cos\left(\frac{u - v}{2}\right) \\
\cos(u) - \cos(v) &= -2 \sin\left(\frac{u + v}{2}\right) \sin\left(\frac{u - v}{2}\right)
\end{align*}
\]
This week, waves. Next Monday, a “glimpse of the future” with Persans. Also, preliminary lab books due on Monday, but I need to figure out grading time given traveling.

**The Wave Concept**

Waves “transfer energy through local motion.” The motion can be “transverse” or “longitudinal” and the energy transfer can be in one, two, or three dimensions. Example of a 1D transverse wave is the stretched string; 2D transverse wave water surface ripples; 3D longitudinal wave is sound in air. Seismic waves are 3D, both transverse and longitudinal. Light is 3D, transverse. A wave may, or may not, be “periodic” or “pulsed.”

** Mathematical Description: General and Sinusoidal**

Illustrate with transverse waves in one dimension, i.e. the stretched string. The transverse dimension of the string is $y$, position along the string is $x$, so the description of the wave is the function $y(x,t)$. See textbook figures 18-4 and 18-5:

The “shape” of the wave is $f(x)$, but this shape moves down the string at some speed $v$. Write $y(x,0) = f(x)$, so this is the shape at $t = 0$. Some time later, the wave has the same shape, but where $x$ has “moved” to $x' = x - vt$, so $y(x,t) = f(x') = f(x - vt)$. A wave can also move in the negative direction, i.e. $y(x,t) = f(x') = f(x + vt)$. That is,

$$y(x,t) = f(x') = f(x \pm vt) \quad (33)$$

where $v$ is some positive number, called the “speed of the wave.”

So, what is actually moving? Any point on the string moves up and down (i.e., transversely) as the wave moves by. Figure 18-5 shows the motion of some point on the string as the pulse represented by $y(x,t) = f(x - vt)$ moves past it. **Note:** Fig.18-5 plots $y$ as a function of $t$ for fixed $x$. The point rises from zero and then falls back as the pulse passes.

**Extremely important but advanced point:** We almost always describe waves as if the shape $f(x \pm vt)$ is a sine function. This is because any general function $f(x)$ can be built out of a (possibly infinitely long) linear combination of sine functions. The proof of this is part of something called “Fourier Analysis” which we won’t cover. But it is true.

A sine wave is obviously periodic. It is handy to describe a sine wave as follows:

$$y(x,t) = y_m \sin \left[ \frac{2\pi}{\lambda} (x - vt) \right] \quad (34)$$
We call \( \lambda \) the wavelength, for obvious reasons. Note that for fixed \( x \) (say \( x = 0 \)) we have 
\[
y(0, t) = y_m \sin \left( -2\pi vt/\lambda \right)
\]
which means that the transverse oscillations are sinusoidal with period \( T = \lambda/v = 1/f \) where \( f \) is the frequency. We also have \( \lambda f = v \). Sometimes we write \( \nu \) instead of \( f \).  

[Geeky physics joke: Question. What’s new? Answer. \( c/\lambda! \)]

The more common way to write the equation of a sine wave is 
\[
y(x, t) = y_m \sin (kx - \omega t - \phi)
\]
where \( k = 2\pi/\lambda \), \( \omega = 2\pi f \) and \( \phi \) is called the phase constant. See textbook on “phase” and “phase constant.” (Compare two waves with same \( k \) and \( \omega \) but different \( \phi \).)

**Waves on a Stretched String: The Wave Equation**

What is the “equation of motion” for a wave? Let’s get it from \( F = ma \) on a string.

The motion is up and down, so analyze the forces in the vertical direction on a small piece of string at fixed \( x \). We assume small amplitude waves so \( \delta x \) can represent the length of the small piece. See Figure 18-10. If \( F \) is the tension in the string, then 
\[
\sum F_y = F \sin \theta_2 - F \sin \theta_1 = (\delta m) a_y
\]
Put \( \delta m = \mu \delta x \) for linear mass density \( \mu \). Obviously \( a_y = \ddot{y} = \partial^2 y/\partial t^2 \) since we are working at fixed \( x \). Also, for small deflections, \( \theta \ll 1 \) so \( \sin \theta \approx \theta \approx \tan \theta = \text{slope} = \partial y/\partial x \). So,
\[
F(\sin \theta_2 - \sin \theta_1) \approx F \left( \frac{\partial y}{\partial x} \bigg|_2 - \frac{\partial y}{\partial x} \bigg|_1 \right) = \mu \delta x \frac{\partial^2 y}{\partial t^2} \Rightarrow \frac{1}{\delta x} \left( \frac{\partial y}{\partial x} \bigg|_2 - \frac{\partial y}{\partial x} \bigg|_1 \right) = \frac{\mu}{F} \frac{\partial^2 y}{\partial t^2}
\]
The left side of the last equation is just \( \partial^2 y/\partial x^2 \) as \( \delta x \to 0 \). So, defining \( 1/v^2 \equiv \mu/F \),
\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}
\]
which is called the “wave equation” because its solution is, in fact, a wave. That is, a function of the form Eq. 33 solves this differential equation for any function \( f(x) \). This is simple to prove. All you need is the chain rule.

Realize that Newton’s Second Law has given us a formula for the speed \( v \) of the wave in terms of physical parameters, in this case the tension \( F \) and density \( \mu \) of the string.

**Practice Exercises**

(1) Write the wave form \( y(x, t) = \frac{1}{2} y_m \left[ e^{i(kx - \omega t)} + e^{-i(kx - \omega t)} \right] \) in the standard way shown in Eq. 35 and determine the phase constant \( \phi \).

(2) See multiple choice question 7 in Chapter 18. Find which of the following wave forms solves the wave equation, Eq. 36, for \( v = 1 \). Show this by taking the derivatives, and by showing which can be written in the form Eq. 33. (You might take a look at the “Principle of Superposition”, Sec.18-7, which we will cover on Thursday.)

(A) \( y(x, t) = x^2 - t^2 \)

(B) \( y(x, t) = \sin x^2 \sin t \)

(C) \( y(x, t) = \log(x^2 - t^2) - \log(x - t) \)

(D) \( y(x, t) = e^x \sin t \)
Homework!
Today: Finish discussion of “waves”, including “standing waves” and “sound”

**Review**

For One-Dimensional, Transverse motion of a stretched string, “\( F = ma \)” leads to

\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \implies y(x, t) = f(x \mp vt)
\]

where \( v = \sqrt{\frac{F}{\mu}} \)

and \( F \) is the string tension and \( \mu \) is the mass per unit length. The (partial) differential equation is called the “wave equation” and its solution is a “wave” moving to the right (−) or left (+) with speed \( v \). Although we derived it for the stretched string, the wave equation turns out to be everywhere in nature.

**Principle of Superposition**

The sum of two waves is also a wave. This is called the “Principle of Superposition” and is easy to prove. It is true because the wave equation is a linear differential equation.

Let \( f_1(x \mp vt) \) and \( f_2(x \mp vt) \) both be waves, i.e. they each solve the wave equation. We claim that \( y(x, t) = f_1 + f_2 \) is also a wave. So let’s plug it in and see:

\[
\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f_1}{\partial t^2} + \frac{1}{v^2} \frac{\partial^2 f_2}{\partial t^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}
\]

Yes, it works. If \( f_1 \) and \( f_2 \) both solve the wave equation, then so does \( y = f_1 + f_2 \). We say that the two waves “interfere” with each other.

A common manifestation of superposition, i.e. wave interference, is “beats.” This happens when adding two sine waves of nearly the same, but not identical, frequencies. See Fig.19-13. The resulting wave has “two” frequencies. One is the average of the two original frequencies, but the other is an “envelope” that modifies the amplitude of the overall wave. You have probably heard beats from sound waves.

**Standing Waves**

Now suppose we interfere two sine waves of the same speed, frequency, and (therefore) wavelength, but one is moving to the right and the other is moving to the left. That is

\[
\begin{align*}
y_1(x, t) &= y_m \sin(kx - \omega t) \\
y_1(x, t) &= y_m \sin(kx + \omega t) \\
\Rightarrow y(x, t) &= y_1(x, t) + y_2(x, t) \\
&= [2y_m \cos(\omega t)] \sin(kx)
\end{align*}
\]

where the last equality follows from a trig identity. This is a very peculiar wave! It has zero velocity (i.e. \( kx = k(x - 0t) \)) but an amplitude that varies in time with angular frequency \( \omega \). It is called a “standing wave.”
This becomes a little clearer when we watch it happen. See Fig.18-17:

Something important happens when we set up standing waves on a stretched string that is fixed at both ends, like a guitar string. As shown in Fig.18-20, we can only have standing waves that “fit” into the distance $L$ between the two fixed ends. That is

$$L = \frac{n \lambda}{2} \Rightarrow \lambda_n = \frac{2L}{n} \Rightarrow f_n = \frac{n}{2L} \frac{v}{n}$$

That is, frequencies have to be integer multiples of a “fundamental” frequency $v/2L$. This is the basis of the harmonic sound of nearly all string and wind instruments.

**Sound Waves**

Sound waves are pressure waves. A disturbance compresses some continuous medium, and the compression propagates.

We expect to come up with a differential equation for the behavior of pressure $p(x,t)$ in a medium, starting with $F = ma$, and that this differential equation would in fact be the wave equation. This is of course true, but we won’t do it here. This wave equation would involve the medium properties, like density $\rho_0$ and bulk modulus $B$. (Note that the density would also fluctuate for a compressional, i.e. longitudinal, wave, so we talk about the mean density $\rho_0$ of the medium.) All of the phenomena associated with waves, like interference, beats, and harmonic frequencies, are readily observed with sound.

It is interesting to consider the “real world” values for sound waves, in terms of their speed and intensity. See Tables 19-1 and 19-2, and Figure 19-5 in your textbook.

**Practice Exercises**

1. Consider two sine waves of the same velocity, but slightly different angular frequencies $\omega$ and $\omega + \Delta \omega$. Find the “beat frequency” which modifies the amplitude of the resultant wave. See the trig identities we used for the practice problems on coupled harmonic oscillators.

2. Use dimensional analysis to come up with an expression for the speed of sound, in terms of the bulk modulus $B$ and density $\rho_0$ of the medium. Recall from “How to look at a fluid” that $B \equiv -\Delta p/(\Delta V/V)$. Compare your answer to Eq.(19-14) in your textbook.
Homework! Turn in preliminary lab books on Monday!
Thanks to Peter for covering class on Monday!

The topic for today is “Lorentz Transformations”, which are the equations that relate measurements of space and time between two different reference frames. They are the mathematical formulation of Special Relativity. There is much in Special Relativity which we will not cover (like velocity addition, relativistic energy and momentum, four-vector notation, relativistic invariants, etc...) but you will see this material in later courses.

**Newton+Maxwell=Einstein**

Newton’s Laws are formulated assuming that “time” is universal. So, take two observers, one \((x')\) moving at speed \(u\) relative to the other \((x)\). Then \(x' = x - ut\). If something moves with speed \(v = dx/dt\) in the \(x\) frame, then it moves with speed \(v' = dx'/dt = v - u\) in the \(x'\) frame. You know this. If you are in moving car, and watch a ball thrown on the ground in the direction you are moving, then it appears to move much more slowly to you than to the person who threw it.

Maxwell delivered a set of equations that described electricity and magnetism. These equations predict the existence of light, because they can be used to derive a wave equation, where the wave speed is \(1/\sqrt{\varepsilon_0\mu_0} \equiv c\), the “speed of light.” Maxwell didn’t know from Newton. Nothing in these equations has anything to do with reference frames. Everyone should measure the same speed of light \(c\), regardless of their relative speeds.

So, Maxwell and Newton are inconsistent, and one or both of them must be wrong. Einstein realized that the problem was that Newton assumed that time was universal. Indeed the time for the \(x'\) observer, call it \(t'\), is not the same as the time \(t\) for the \(x\) observer. That’s really all there is to Special Relativity.

**Intervals**

What are the observable consequences of \(c\) being the same for all observers? Consider the answer by studying “intervals” of space \(\Delta x\) and time \(\Delta t\). See Figs 20-4 and 20-5, below.

![Diagram of light beam bouncing between mirrors](image)

On the left, a light beam bounces up and down between two mirrors in a frame called \(S'\). It travels a distance \(2L_0\) in a time \(\Delta t' \equiv \Delta t_0 = 2L_0/c\). This is a kind of clock.

Let the frame \(S'\) move to the right with a speed \(u\) relative to a frame \(S\). What is the time interval \(\Delta t\) to an observer in \(S'\)? The clock moves to the right an amount \(u\Delta t\) in this time.
interval, and the light travels a distance \(2L\) where \(L^2 = L_0^2 + (u\Delta t/2)^2\). That is
\[
(u\Delta t)^2 = (2L)^2 - (2L_0)^2 = (c\Delta t)^2 - (c\Delta t_0)^2
\]
Divide through by \(c^2\) and solve for \(\Delta t\) to find
\[
\Delta t = \gamma \Delta t_0 \quad \text{where} \quad \gamma \equiv \frac{1}{\sqrt{1 - u^2/c^2}}
\] (37)
Obviously \(\gamma > 1\). Therefore, the time interval is longer for the observer “on the ground.” We say the clock ticks more slowly for the observer who “is moving.” This phenomenon is known as “time dilation.” There is a related phenomenon called “length contraction.” See your textbook for the discussion.

The Lorentz Transformation

The equations that give you \(x'\) and \(t'\) in terms of \(x\) and \(t\) are called the Lorentz Transformation. They are not hard to derive, but we won’t do it here. Just imagine two reference frames, one moving past the other, and let them coincide origins at their respective zeros of time. That is, \((x', t') = (0, 0)\) when \((x, t) = (0, 0)\). Then, let a pulse of light move out from the origin, and require that both reference frames see it move at a speed \(c\). If the relative speed of the reference frames is \(u\) then you find that
\[
x' = \gamma(x - ut)
\] (39)
\[
t' = \gamma(t - ux/c^2)
\] (40)
You can of course solve these for \(x\) and \(t\) in terms of \(x'\) and \(t'\), but you know that the answer has to be the same as switching \(u\) to \(-u\), right? That is,
\[
x = \gamma(x' + ut')
\] (41)
\[
t = \gamma(t' + ux'/c^2)
\] (42)
It is easy to prove that for the quantity \(\Delta s\) called the “invariant interval”,
\[
\Delta s^2 \equiv (\Delta t)^2 - (\Delta x)^2 = (\Delta t')^2 - (\Delta x')^2
\]
That is, \(\Delta s\) is the same for all reference frames.

Natural Units

Why should we bother to carry the \(c\) around all the time? Think of it as a conversion factor between space and time. That is, measure time in seconds, but measure distance in light-seconds, which is the distance light travels in a second. This in fact puts the Lorentz Transformation into an interesting form. We have
\[
x' = \gamma t - \gamma ut
\]
\[
t' = \gamma t - \gamma ux
\]
But \(\gamma^2 - (\gamma u)^2 - 1\), so we can write \(\gamma = \cosh \alpha\) for some value of \(\alpha\), which gives \(\sinh \alpha = \gamma u\). In this case, the Lorentz Transformation is written as
\[
x = (\cosh \alpha) x' + (\sinh \alpha) t'
\]
\[
t = (\sinh \alpha) x' + (\cosh \alpha) t'
\]
This looks rather like a “rotation”, but using hyperbolic cosines and sines instead of circular (i.e. “normal”) sines and cosines.
Exercise

This exercise will be graded as a homework problem! Please hand it in.

The “Lorentz Transformation” given by Equations 39, 40, 41, and 42 change reference frames in special relativity. In this activity, you will go through the mechanics of a Lorentz transformation graphically. You are encouraged to check your answers using the equations, but by working through this on a graph, you will get a better physical feel for what these transformations are really about.

Imagine that your friend David is on a new space shuttle heading for α-Centauri. Your reference frame is designated by \((x, t)\) and David’s by \((x', t')\). Answer the following questions using the accompanying graph, which gives you a \((x, t)\) coordinate axis. Get numerical values as best you can. The graph includes a dashed line (“c”) showing \(x = ct\). Also shown are the \(x'\) and \(t'\) axes, that is, the lines along which \(t' = 0\) and \(x' = 0\), respectively. The black dots show the points \((x', t') = (0, 0), (0, 1), (1, 0), (0, -1),\) and \((-1, 0)\).

1. What is David’s velocity in your reference frame? You will find it useful to draw the “world line” for a particle with a fixed value of \(x'\).

2. What is your velocity in David’s reference frame?

3. In David’s reference frame, he measures the time between two events to be 1 sec. What do you say the time is? You can let the events be at the same place in David’s frame.

4. In your reference frame, you measure the time between two events to be 1 sec. What does David say the time is? To do this, draw a line parallel to David’s \(x'\) axis.

5. In your reference frame, you measure the distance between simultaneous events to be 1 light-sec. What does David say the distance is?

6. In David’s reference frame, he measures the distance between simultaneous events to be 1 light-sec. What do you say the distance is? To do this, you need to draw a line parallel to David’s \(t'\) axis.

7. Consider an object moving between two points. The first point is at \(x = 0\) and \(t = 0\). The second point is at \(x = 2\) and \(t = 1/2\). How fast is this object moving in your reference frame?

8. When and where are these points in David’s reference frame?

9. What does this tell you about an object going faster than the speed of light?
Imagine that your friend David is on a new space shuttle heading for α-Centauri. Your reference frame is designated by \((x, t)\) and David’s by \((x', t')\). Answer the following questions using the accompanying graph, which gives you a \((x, t)\) coordinate axis. Get numerical values as best you can. The graph includes a dashed line (“\(c\)”) showing \(x = ct\). Also shown are the \(x'\) and \(t'\) axes, that is, the lines along which \(t' = 0\) and \(x' = 0\), respectively. The black dots show the points \((x', t') = (0, 0), (0, 1), (1, 0), (0, -1), \) and \((-1, 0)\).

1. What is David’s velocity in your reference frame? You will find it useful to draw the “world line” for a particle with a fixed value of \(x'\).

Pick a fixed point in David’s frame, say \(x' = 0\). For \(\Delta t = 1\), we find \(\Delta x = 1/2\), so \(u = \Delta x/\Delta t = 1/2\). This gives \(\gamma = 1/\sqrt{1 - (1/2)^2} = 2/\sqrt{3} = 1.15 = 1/0.866\).

2. What is your velocity in David’s reference frame?

Now pick \(x = 0\), and follow it along for \(\Delta t' = 1\). One gets back to \(x = 0\) for \(\Delta x' = -1/2\). So, \(u = \Delta x'/\Delta t' = -1/2\).

3. In David’s reference frame, he measures the time between two events to be 1 sec. What do you say the time is? You can let the events be at the same place in David’s frame.

Pick the two events at \(x' = 0\) and with \(t' = 0\) (which is also \(t = 0\)) and \(t' = +1\). Read it off the \(t'\) axis to get \(t' \approx 1.2(= 1.15)\).

4. In your reference frame, you measure the time between two events to be 1 sec. What does David say the time is? To do this, draw a line parallel to David’s \(x'\) axis.

Now stick at \(x = 0\) and look at \(t = 0\) (which is also \(t' = 0\)) and \(t = 1\). Draw a line parallel to David’s \(x'\) axis and see that the \(t = 1\) point intersects David’s \(t'\) axis at a little more than \(t' = 1\), say at \(t' \approx 1.2(= 1.15)\).

5. In your reference frame, you measure the distance between simultaneous events to be 1 light-sec. What does David say the distance is?

Pick the two points at \(t = 0\) (“simultaneous”) and \(x = 0\) and \(x = 1\). Using the same approach as above, it is easy to see that the \((x, t) = (1, 0)\) has a value of \(x'\) “a little large than 1”, that is, \(x' \approx 1.2(= 1.15)\).

Don’t be confused by the “length contraction” effect. You are the person measuring a “length” here, not David, because you’re the one making the measurement using simultaneous events. Your length measurement is “contracted” relative to David’s.

This is easy to see mathematically using the invariant relativistic interval

\[
\Delta s^2 = (\Delta t)^2 - (\Delta x)^2 = (\Delta t')^2 - (\Delta x')^2
\]
which must have the same value in all reference frames. Therefore, if $\Delta t = 0$,

$$(\Delta x')^2 = (\Delta x)^2 + (\Delta t')^2$$

and so $\Delta x'$ has to be larger than $\Delta x$.

6. In David’s reference frame, he measures the distance between simultaneous events to be 1 light-sec. What do you say the distance is? To do this, you need to draw a line parallel to David’s $t'$ axis.

Look at the points on the plot at $(x, t) = (x', t') = (0, 0)$ and $(x', t') = (1, 0)$. These are simultaneous in David’s frame. Just read the $x$ value off the graph, and get $x = 1.15$.

7. Consider an object moving between two points. The first point is at $x = 0$ and $t = 0$. The second point is at $x = 2$ and $t = 1/2$. How fast is this object moving in your reference frame?

The speed of this object is $\Delta x/\Delta t = 4$, that is, four times the speed of light.

8. When and where are these points in David’s reference frame?

The later time point is close to $(x', t') = (2, -0.5)$.

9. What does this tell you about an object going faster than the speed of light?

The object is moving backwards in time in David’s frame.
Today we start the last main topic of the course, namely “Thermodynamics and Statistical Mechanics.” Some of this may be familiar from your Chemistry class.

**Temperature: A Mysterious Concept**

You all think you know what “temperature” is, but be careful. It is complicated, especially when we want to think about it scientifically. Its definition relies on the concept of “thermal equilibrium.” There is something called the *zeroth law of thermodynamics* which states

*If systems A and B are each in thermal equilibrium with a third system C, then A and B are in thermal equilibrium with each other.*

Not much of a law, since we haven’t defined “thermal equilibrium,” but at least it lets us use something called a “thermometer” to measure something’s temperature. There will be more insight into the meaning of temperature when we study so-called statistical mechanics by considering bulk materials at the molecular scale.

There are common temperature scales used in everyday life, although neither is used much by physicists, who favor a third “absolute” temperature scale. The two ordinary scales are Fahrenheit \((T_F, \text{ defined by the freezing point of brine and body temperature})\) and Celsius \((T_C, \text{ defined by the freezing and boiling points of pure water})\). The Kelvin \((T)\) scale is based on “absolute zero” and the so-called “triple point” of pure water. We have

\[
T_C = T - 273.15
\]

\[
T_F = \frac{9}{5} T_C + 32
\]

The idea of “absolute zero” will be clearer after we learn some things about gases. By extrapolating down from “room temperature” it seems that pretty much all gases would have zero volume if we cooled them to \(T = 0\).

**Thermal Expansion**

Most solids expand when they get hot. The reason is subtle. Figs 21-9 and 21-12:

Higher temp means faster molecular motion. (Later!) Solids are like atoms connected by “anharmonic” springs, and average position shifts to larger separations as the energy increases. So, solids expand.

The effect is rather linear and not large for typical temperatures. For solids, we write

\[
\Delta L = \alpha L \Delta T
\]

where \(\alpha\) is called the *coefficient of linear expansion* and expresses the fractional change in length \((\Delta L/L)\) per degree of temperature change, and varies from solid to solid. (See Table 21-3.) Since the fractional change is small, we expect \(\Delta A = 2\alpha A \Delta T\) for expansion of surface
area, and $\Delta V = 3\alpha V \Delta T$ for volume. (Remember your Taylor expansions!) Note that fluids (like water!) are more complicated than solids.

**The Ideal Gas**

A chemist will tell you that most gases at “normal” pressures and temperatures, follow

$$pV = nRT$$

(44)

where $p$ is the pressure, $T$ is the (absolute) temperature, $V$ is the volume of the container, and $R$ is a constant. Also, $n$ is the number of “moles” of gas. (What’s a mole?) Let us try to understand this as physicists. In the process, we will get a better feeling for what “temperature” actually means.

See Fig.22-2. Put $N$ gas molecules each of mass $m$ in a cubic box of side length $L$. Molecule has $x$ velocity $v_x$. It bounces off the wall and incurs momentum change $2mv_x$. The time between bounces is $2L/v_x$ so $F_x(2L/v_x) = 2mv_x$. The pressure on the wall at $x = L$ is the sum of the forces of all the molecules, divided by the area of the wall. That is

$$p = \frac{1}{L^2} \sum F_x = \frac{1}{L^2} \sum \frac{mv_x^2}{L} = \frac{Nm}{V} \left( \frac{1}{N} \sum v_x^2 \right)$$

where we recognize that the volume of the cube is $V = L^3$. Now the quantity in parentheses is the average value of $v_x^2$, written $\langle v_x^2 \rangle$. There is nothing special about the $x$ direction, and the number $N$ is very large, so we expect that $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle = \frac{1}{3} \langle v^2 \rangle$ where we call $\langle v^2 \rangle$ the “mean square velocity.” Since all molecules have the same mass, $\langle K \rangle = \frac{1}{2}m\langle v^2 \rangle$ is the average kinetic energy of the molecules. So, we can now write $pV = N\frac{2}{3} \langle K \rangle$.

Back to what chemists call a “mole.” This is the number of molecules divided by Avogadro’s number $N_A = 6.02 \times 10^{23}$. That is, $n = N/N_A$. Therefore, we recover Eq. 44 with

$$pV = nRT \quad \text{where} \quad \langle K \rangle \equiv \frac{3}{2} kT \quad \text{and} \quad R \equiv kN_A$$

This show that temperature (for the ideal gas) measures the average kinetic energy of the molecules! Experimentally, we find $k = 1.38 \times 10^{-23} \text{J/K}$, which we call “Boltzmann’s constant.” Physicists don’t usually use moles and $N_A$ and instead write the ideal gas law as

$$pV = NkT$$

(45)

We will take this molecular approach to thermodynamics farther on Thursday.

**Practice Exercises**

(1) Determine the temperature at which the Fahrenheit and Celsius scales coincide, that is, give the same value for the temperature. Have you ever been outside on a day or night when this was the air temperature?

(2) A large flat slab of metal has a hole in the middle of it. The slab is warmed by increasing its temperature. Does the diameter of the hole increase, decrease, remain the same, or depend on the thermal expansion coefficient of the particular metal?

(3) Explain why our definition $\langle K \rangle \equiv \frac{3}{2} kT$ for the temperature of an ideal gas, would not make sense if we wrote it in terms of $T_F$ or $T_C$, instead of absolute temperature $T$. 
Today: “How to build a star.” (I was going to do Maxwell-Boltzmann distribution for molecular speeds, but this will be more fun.)

Note: I will be traveling before and after Thanksgiving, so Monday, Nov.19 (“Heat and the First Law”) will be covered by John Cummings, then Monday, Nov.26 (“Entropy”) will be covered by Angel Garcia, and then Exam #3 on Thursday, Nov.29 (on material through Monday, Nov.19).

How to Build a Star: Hydrostatics of really big ball of gas

Start with the balance of pressure against gravitation. Then use ideal gas law to relate to temperature. (Will assume that stars like the sun have uniform density.) Finish with the behavior of white dwarf stars, aka “degenerate Fermi gas.”

More reading (in order of increasing sophistication):


Begin. First, set up the notation. Build our star with shells of radius \( r \). Total mass of star is \( M \) and radius is \( R \). Density is \( \rho = M/(4\pi R^3/3) \) assumed to be independent of \( r \). Write \( \tilde{M}(r) \) be the mass enclosed at radius \( r \), i.e. \( \tilde{M}(R) = \rho(4\pi R^3/3) = M(r/R)^3 \).

Now, balance pressure \( p(r) \) against gravity. Take a small piece of area \( A \), thickness \( dr \) and mass \( dm = \rho Adr \), at radius \( r \). Outward force from difference in pressure must equal the gravitational force from mass inside radius \( r \). (Recall “shell theorems.”) Write it out:

\[
pdA - (p + dp)A = G\frac{\tilde{M}(r)dm}{r^2} = G\frac{\tilde{M}(r)\rho Adr}{r^2} \text{ or,} \]

\[
\frac{dp}{dr} = -G\frac{\tilde{M}(r)\rho(r)}{r^2} \quad (46)
\]

This equation expresses “hydrostatic equilibrium” for a large, spherical self-gravitating object like a star (or a planet, or a moon). It is written here in a way that holds for any density function \( \rho(r) \), but we will be taking \( \rho \) to be a constant.

Let’s use this to determine the pressure \( p_C \) at the center of a star, i.e. \( r = 0 \). Make the replacements for constant density in Eq. 46 to get

\[
\frac{dp}{dr} = -GM \left( \frac{r}{R} \right)^3 \frac{M}{4\pi R^3/3} \frac{1}{r^2} = -\frac{3GM^2}{4\pi R^6} r \quad \implies \quad p(r) = p_C - \frac{3GM^2}{8\pi R^6} r^2
\]

At the surface of the star, the pressure is zero “by definition”, i.e. \( p(R) = 0 \). Therefore,

\[
p_C = \frac{3GM^2}{8\pi R^4} \quad (47)
\]

Next let us try to estimate the temperature at the center of a star. This requires us to know the “equation of state” which relates the gas pressure and density to the temperature, for...
the matter that makes up the star. Understanding the equation of state is a big deal, and an area of active research interest. For now, we will use an equation of state based on an ideal gas, that is Eq. 45, namely \( pV = NkT \).

If we assume that the star is made of \( N \) identical particles with mass \( m \), then \( \rho = Nm/V \) and \( p = \rho kT/m = (3/4\pi)(M/m)kT/R^3 \). In fact, stars are hot, so hot that matter is a “plasma” of free nuclei and electrons. The nuclei are from mostly hydrogen, about 25% helium, and a few percent of heavier elements. For now, though, it is close enough to assume the star is 100% hydrogen atoms. Therefore, the temperature at the center of a star is \( T_C = \frac{1}{k} \frac{4\pi R^3 m}{M} \frac{GmM}{2R} \) (48).

Can you see why we would not be able to get this answer from dimensional analysis?

**The White Dwarf Star**

We found some properties of stars assuming they consisted of an ideal gas, but the assumptions used to get Eq. 45 don’t always hold. Instead, we need another equation of state. In deriving Eq. 45 we wrote \( F_x = pxv_x/L \), and that is fine. For \( N \) particles, this again gives a pressure \( P = NF_x/L^2 = Npv/V \). (We temporarily switch to \( P \) to avoid confusion with momentum. Also, we are making rough estimate, so put \( px \) and \( v_x \) to \( p \) and \( v \).) Our concern will be with electrons, so write an electron density \( n_e = N/V \) in which case \( P = n_e pv \).

At very high densities, the ideal gas law assumptions are violated. The electrons are so close together, they start to violate the Pauli Exclusion Principle. The distance between them is \( d \approx \frac{1}{n_e^{1/3}} \) and the Heisenberg Uncertainty Principle says that \( pd \approx \hbar \) so that \( p \approx \hbar n_e^{1/3} \). Putting \( v = p/m_e \) and switching back to \( p \) for pressure, the equation of state becomes \( p = n_e \left( \hbar n_e^{1/3} \right) \left( \hbar n_e^{1/3} / m_e \right) = \frac{\hbar^2}{m_e} n_e^{5/3} \sim \frac{\hbar^2}{m_e} N^{5/3} = \frac{\hbar^2}{m_e} \left( \frac{\rho}{m} \right)^{5/3} \) (49).

where we make the (somewhat incorrect) assumption that the star is still made up of hydrogen atoms with mass \( m \). This is an odd sort of equation of state. Combined with Eq 47 it leads to a star whose radius decreases with mass like \( 1/M^{1/3} \). See the practice exercise.

As the mass increases, and the radius decreases, the electrons are more and more confined until they are moving as fast as they can, i.e. \( v = c \). The equation of state becomes \( p = n_e \left( \hbar n_e^{1/3} \right) c = \hbar c n_e^{4/3} \sim \hbar c N^{4/3} = \hbar c \left( \frac{\rho}{m} \right)^{4/3} \) (50).

Combined this with Eq 47 gives an equation where the radius drops out! One solves for a mass that is the highest possible mass of a white dwarf star. Doing a more careful job leads to a value of 1.44 solar masses. This is the “Chandrasekhar limit” after the physicist who first derived it. What do you suppose would happen for a star of mass larger than this?

**Practice Exercises**

1. Use Eq. 47 to estimate the central pressure of the Sun. Express it in units of atmospheres. Use values which you can look up in Appendix C of your textbook. Then, use Eq. 48 to estimate the central temperature.

2. Recall that temperature is related to the average kinetic energy of the gas particles. What is the average kinetic energy of the hydrogen atoms in the center of the Sun? Does this give you a hint as to the source of the Sun’s energy?

3. Show that Eq. 49 leads to a “star” with radius \( R \) and mass \( M \) where \( R \propto 1/M^{1/3} \).
Thermodynamics is the only physical theory of universal content which, within the framework of the applicability of its basic concepts, I am convinced will never be overthrown. – Albert Einstein

**First law of thermodynamics** We all as physicist have great faith in the law of conservation of energy. Yet we seem to see it violated every day constantly. If I slide a book across the table to a friend, the books slows and stops, hopefully close enough to my friend for him to reach it. Has energy been lost? Where did it go? J.P. Joule, the English physicist, carried out careful measurements around 1850 to show that mechanical energy was converted into heat. Using falling weights to drive mechanical agitators in tanks of water, and carefully measuring the temperature of the water tanks, he showed that the mechanical energy was showing up as heat in the tanks. Once we understand that heat is another form of energy, this allows us to write the law of energy conservation for a thermodynamic system as

\[ Q + W = \Delta E_{\text{int}} \]  

where \( Q \) is the heat that flows into the system, \( W \) is the work done on the system, and \( E_{\text{int}} \) is the internal energy of the system.

**Internal energy** What is this internal energy? It may seem like a cheat at first, to save conservation of energy, but let’s take advantage of what we know about ideal gasses to understand it better. You know that the average kinetic energy of an ideal gas molecule is

\[ \langle K \rangle = \frac{3}{2} kT, \]  

and so, for a sample of \( N \) molecules we see

\[ E_{\text{int}} = N\left(\frac{3}{2} kT\right). \]  

So the internal energy is determined by temperature, at least for an ideal gas, and is the total kinetic energy of the gas molecules.

It turns out, as derived by J.C. Maxwell, that the theorem of equipartition of energy says that the energy of a molecule is shared equally among all of its independent degrees of freedom – \( \frac{1}{2} kT \) for each degree of freedom. If a molecule has more degrees of freedom, such as a diatomic molecule that has 2 rotational degrees of freedom, the internal energy will be given by

\[ E_{\text{int}} = N\left(\frac{5}{2} kT\right) \]  

at low temperatures (like room temperature for most molecules). At higher temperatures, vibrational modes become available and the internal energy becomes

\[ E_{\text{int}} = N\left(\frac{7}{2} kT\right) \]  

.  

**Heat capacity** The *heat capacity* of a body is defined to be

\[ C = \frac{Q}{\Delta T} \]  

\[ Q + W = \Delta E_{\text{int}} \]  

where \( Q \) is the heat that flows into the system, \( W \) is the work done on the system, and \( E_{\text{int}} \) is the internal energy of the system.
where $Q$ is the heat energy transferred to the body and $\Delta T$ is the resulting temperature change. This is often normalized to a unit mass to obtain a property of the material, rather than a particular object, giving

$$c = \frac{C}{m} = \frac{Q}{m\Delta T} \quad (57)$$

where $c$ is called the specific heat of the material.

The term capacity is unfortunate. A more apt analogy might be cross sectional area of a water tank. A large area (specific heat) means a given amount of water (heat) will raise the level (temperature) very little, while a small area (specific heat) means the level (temperature) will go up a lot.

**Work on an ideal gas** The work $W$ done on a thermodynamic system can be studied by considering a cylinder and piston with cross sectional area $A$ containing an ideal gas at a pressure $p$. The work done on the gas as we move the piston is

$$W = \int F_x \, dx = \int (-pA) \, dx \quad (58)$$

or

$$W = -\int p \, dV \quad (59)$$

This suggests a very useful way to visualize work done on a thermodynamic system: the area under a $pV$ curve. As the system moves from one state to another it traces a path on the $pV$ diagram and the work done on the system is the area under the curve.

---

There are many ways to get from one point to another on a $pV$ diagram, and not all will use or produce the same amount of work. It is this fact which makes possible cyclic changes to the system in a way that changes heat input into work – a heat engine.

**Practice Exercises**

1. What is the shape of an isotherm (a line of constant temperature) on a $pV$ plot?
2. When I open my dishwasher after the dry cycle, glassware and silverware are dry, but plastic containers are still wet. Why?
3. Which will burn your mouth more, pizza straight from the oven or bread straight from the same oven? Why?
4. I want to have hot coffee, but I have no electricity, so I hook up a paddle wheel to a pulley and 1 kg weight. My cup holds 0.5 l and is at 20°C. The weight quickly reaches terminal velocity. How far must it fall to boil the water?
Last (real) class! Where did the time go??

- Hand back Exam#3 if available.
- Final exam Thurs 13 Dec 3-6pm Ricketts 203 (see sis.rpi.edu)
- No lab on Wednesday. We’ll be available for help on lab books, and in general.
- Thursday: Homework due, Lab books due, class devoted to course review.
- Thanks to John Cummings for class on “First Law of Thermodynamics”
- Thanks to Angel Garcia for class on the “Second Law”; No notes, but there will be one problem on the final exam based on homework from Chapter 24.
- Need to leave time at the end of this class for course evaluation forms.

The Principle of Least Action

What is Physics? I think of it as asking “Why?” and not being satisfied with the answer you get. Physicists are always trying to understand why nature is the way it is, and looking for the simplest answer to that question.

The best answer we have today for “Why?” is “Because it minimizes the action.” This quantity “action” is built from various symmetry principles that are realized differently in different physical systems. A physical system will behave in the way that minimizes the action from the start to the finish of some process.

As you learn more physics, you will see how the action is constructed for mechanical systems, electromagnetic systems, and the role that action plays in the formulation of quantum mechanics. (Try looking up something called “Fermat’s principle” and its applications to optics.) Today, though, we’ll do one simple example, which connects directly onto the physics with which we started this course.

Example: Motion under Constant Acceleration

Consider some object with mass $m$ moving with a constant acceleration $a$. Obviously the force is $F = ma$ and the potential energy function is $U(x) = -max$ for motion in one dimension $x$. Let’s form a quantity $L$ that is a function of $x$ and $\dot{x}$ by

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - U(x) = \frac{1}{2}m\dot{x}^2 + max$$

(Even though $L(x, \dot{x})$, called the “Lagrangian”, looks kind of like the total energy $E$ except for the $-$ sign, it is a very different beast so don’t confuse the two!)

Before we define the action, we need to introduce the concept of a path. For one dimensional mechanical motion, a path (through time) is some way of getting from position $x_1$, starting at time $t_1$, and ending up at position $x_2$ at time $t_2$. There are an infinite number of paths connecting the points $(x_1, t_1)$ and $(x_2, t_2)$.

Here’s the “Why?” question. Why does the object follow the path that it does? We know from our physics studies so far that an object undergoing constant acceleration follows the path $x(t) = \frac{1}{2}at^2 + v_0t + x_0$, but we got there from Newton’s Laws. Newton was a smart
guy, but he was only talking about motion, and not some vast underlying principle. Can we get a better answer to our “Why?” question?

The better answer is this: The object follows the path \( x(t) \) which gives the smallest value for the action which is the integral of \( L(x, \dot{x}) \) over the path:

\[
S[x(t)] = \int_{t_1}^{t_2} L(x, \dot{x}) \, dt
\]  

The notation \( S[x(t)] \) indicates that you get a different value of \( S \) for a different path \( x(t) \), but of course there is no dependence any more on \( x \) since it disappears after the integral is carried out.

Let’s work out the integral for the path we believe is correct, i.e. \( x(t) = \frac{1}{2}at^2 + v_0t + x_0 \). You will carry this out for other paths in the practice exercises, and you’ll find that your answer is larger than what we get here, assuming the Principle of Least Action is correct.

Measure \( x \) in meters and \( t \) in seconds, and start the object out at \((x_1, t_1) = (0, 0)\) and go through to \((x_2, t_2) = (1, 1)\). The “right” path that connects these two points is quadratic in time, so \( x(t) = t^2 \), i.e. \( a = 2 \). Equation 60 becomes \( L(x, \dot{x}) = 2mt^2 + 2mt^2 = 4mt^2 \). So,

\[
S[x(t)] = \int_0^1 4mt^2 \, dt = \frac{4}{3}m
\]

There, you’ve just calculated the action for the path \( x(t) = t^2 \). (The units of action are Energy \( \times \) time = \( ML^2/T \), so we really have \( S = (4m/3) \) m\(^2\)/sec in this example.)

This value itself isn’t so important. What’s important is that this should be the smallest value you can get for \( S \) for any path \( x(t) \) that connects \((x_1, t_1) = (0, 0)\) and \((x_2, t_2) = (1, 1)\).

In later physics courses, you will use something called “Calculus of variations” to show that minimizing the action is equivalent to writing \( m\ddot{x} = -dU/dx = F \), that is, Newton’s Second Law. Even Maxwell’s Equations can be derived by minimizing the action! The Principle of Least Action has a deeper meaning than Newton or Maxwell.

**Practice Exercise**

Try some other path that connects \((x_1, t_1) = (0, 0)\) and \((x_2, t_2) = (1, 1)\), and show that the action \( S[x(t)] \) is something larger than \( 4m/3 \). Any path \( x(t) = t^n \) will do, if \( n \geq 1 \). (What’s wrong with \( x(t) = \sqrt{t} \)?) You might want to show that

\[
S[(t^n)] = m \left[ \frac{n^2}{2(2n-1)} + \frac{2}{n+1} \right]
\]

and plot \( S \) for different values of \( n \).

Another path that you can try is \( x(t) = \sin(\pi t/2) \). The integral is only a little tricky, but you can find it in your Schaum’s outline. The answer works out to something involving terms with powers of \( \pi \), but again the result is greater than \( 4m/3 \).

Can you think of other paths to try?
Final Review

The final exam will be on Thursday 13 December, from 3-6pm in Ricketts 203 (see sis.rpi.edu). The format is 20 multiple choice problems worth two points each (no partial credit) and six short problems, worth ten points each.

Here’s a quick summary of what we learned this year.

- **General Topics**
  - Dimensional analysis
  - Uniform circular motion
  - Einstein’s equivalence principle
  - Coupled oscillations and the eigenvalue problem
  - Introduction to special relativity and Lorentz Transformations
  - The Principle of Least Action as a foundation for physical law

- **Mathematics topics**
  - Vector algebra: Dot products and cross products
  - Taylor expansions; Complex numbers and imaginary exponents
  - Line integrals and surface integrals; Vector fields and the divergence theorem

- $F = ma$ as the “equation of motion”
  - Physics in terms of differential equations
  - Motion in two and three dimensions (vectors)
  - Constants of the motion, e.g. $\ell$ and $E$
  - Frictional forces; Drag forces, proportional to velocity
  - The simple pendulum

- **Work and potential energy**
  - Work as $W = \int \vec{F} \cdot d\vec{s}$; Potential energy as $F = -\frac{dU(x)}{dx}$
  - “Path Independence” and “Conservative Forces”
  - Various typical potential energy functions

- **Gravitation and orbits**
  - Newton’s law of gravity; Gravity near the surface of the Earth
  - The shell theorems
  - Gravitational potential energy
  - Circular orbits; The Dark Matter problem
  - Elliptical orbits and Kepler’s Laws
• Oscillations in one dimension
  – Harmonic approximation for potential “wells”
  – Simple harmonic motion; Amplitude and phase
  – Damped harmonic motion; Forced oscillations
• Systems of particles and solid objects
  – Momentum and impulse
  – Center of mass; Conservation of momentum
  – Calculating the center of mass for solid objects
  – Rotational kinematics
  – Angular velocity and angular acceleration as vectors
  – Torque, rotational inertia, and \( \tau = I \alpha \)
  – Calculating the rotational inertia of solid objects
  – The parallel axis theorem
  – Angular momentum of a particle and a solid object
  – Conservation of angular momentum
• Fluids
  – Hydrostatics, Pascal’s Principle, and Archimede’s Principle
  – The Continuity Equation and Bernoulli’s Equation
  – The Continuity Equation in terms of vector fields
• Waves
  – Properties and mathematical description of waves
  – Waves on a stretched string; The wave equation
  – Superposition and standing waves
  – Sound waves
• Thermodynamics
  – Temperature, heat, and the First Law
  – Work and heat in closed cycles
  – Entropy and the Second Law
  – Kinetic theory and the Ideal Gas Law
  – Thermal expansion and specific heat
  – Static properties of stars; White dwarf stars