Homework! Turn in preliminary lab books on Monday!
Thanks to Peter for covering class on Monday!

The topic for today is "Lorentz Transformations", which are the equations that relate measurements of space and time between two different reference frames. They are the mathematical formulation of Special Relativity. There is much in Special Relativity which we will *not* cover (like velocity addition, relativistic energy and momentum, four-vector notation, relativistic invariants, etc...) but you will see this material in later courses.

Newton+Maxwell=Einstein

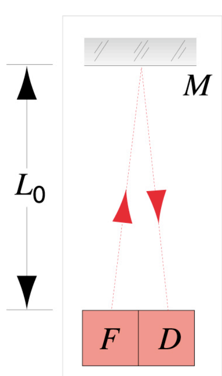
Newton's Laws are formulated assuming that "time" is universal. So, take two observers, one (x') moving at speed u relative to the other (x). Then $x' = x - ut$. If something moves with speed $v = dx/dt$ in the x frame, then it moves with speed $v' = dx'/dt = v - u$ in the x' frame. You know this. If you are in moving car, and watch a ball thrown on the ground in the direction you are moving, then it appears to move much more slowly to you than to the person who threw it.

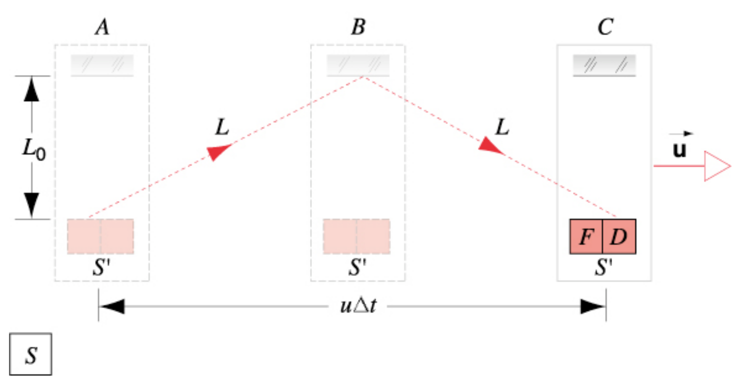
Maxwell delivered a set of equations that described electricity and magnetism. These equations predict the existence of light, because they can be used to derive a wave equation, where the wave speed is $1/\sqrt{\epsilon_0\mu_0} \equiv c$, the "speed of light." Maxwell didn't know from Newton. Nothing in these equations has anything to do with reference frames. Everyone should measure the same speed of light c, regardless of their relative speeds.

So, Maxwell and Newton are inconsistent, and one or both of them must be wrong. Einstein realized that the problem was that Newton assumed that time was universal. Indeed the time for the x' observer, call it t', is not the same as the time t for the x observer. That's really all there is to Special Relativity.

Intervals

What are the observable consequences of c being the same for all observers? Consider the answer by studying "intervals" of space Δx and time Δt. See Figs 20-4 and 20-5, below.





On the left, a light beam bounces up and down between two mirrors in a frame called S'. It travels a distance $2L_0$ in a time $\Delta t' \equiv \Delta t_0 = 2L_0/c$. This is a kind of clock.

Let the frame S' move to the right with a speed u relative to a frame S. What is the time interval Δt to an observer in S? The clock moves to the right an amount $u\Delta t$ in this time interval, and the light travels a distance $2L$ where $L^2 = L_0^2 + (u\Delta t/2)^2$. That is

$$(u\Delta t)^2 = (2L)^2 - (2L_0)^2 = (c\Delta t)^2 - (c\Delta t_0)^2$$

Divide through by c^2 and solve for Δt to find

$$\Delta t = \gamma \Delta t_0 \qquad \text{where} \tag{37}$$

$$\gamma \equiv \frac{1}{\sqrt{1 - u^2/c^2}} \tag{38}$$

Obviously $\gamma > 1$. Therefore, the time interval is longer for the observer "on the ground." We say the clock ticks more slowly for the observer who "is moving." This phenomenon is known as "time dilation." There is a related phenomenon called "length contraction." See your textbook for the discussion.

The Lorentz Transformation

The equations that give you x' and t' in terms of x and t are called the Lorentz Transformation. They are not hard to derive, but we won't do it here. Just imagine two reference frames, one moving past the other, and let them coincide origins at their respective zeros of time. That is, $(x', t') = (0, 0)$ when $(x, t) = (0, 0)$. Then, let a pulse of light move out from the origin, and require that both reference frames see it move at a speed c. If the relative speed of the reference frames is u then you find that

$$x' = \gamma(x - ut) \tag{39}$$

$$t' = \gamma(t - ux/c^2) \tag{40}$$

You can of course solve these for x and t in terms of x' and t', but you know that the answer has to be the same as switching u to $-u$, right? That is,

$$x = \gamma(x' + ut') \tag{41}$$

$$t = \gamma(t' + ux'/c^2) \tag{42}$$

It is easy to prove that for the quantity Δs called the "invariant interval",

$$\Delta s^2 \equiv (\Delta t)^2 - (\Delta x)^2 = (\Delta t')^2 - (\Delta x')^2$$

That is, Δs is the same for all reference frames.

Natural Units

Why should we bother to carry the c around all the time? Think of it as a conversion factor between space and time. That is, measure time in seconds, but measure distance in light-seconds, which is the distance light travels in a second. This in fact puts the Lorentz Transformation into an interesting form. We have

$$\begin{aligned} x' &= \gamma x - \gamma u t \\ t' &= \gamma t - \gamma u x \end{aligned}$$

But $\gamma^2 - (\gamma u)^2 - 1$, so we can write $\gamma = \cosh\alpha$ for some value of α, which gives $\sinh\alpha = \gamma u$. In this case, the Lorentz Transformation is written as

$$\begin{aligned} x &= (\cosh\alpha)x' + (\sinh\alpha)t' \\ t &= (\sinh\alpha)x' + (\cosh\alpha)t' \end{aligned}$$

This looks rather like a "rotation", but using hyperbolic cosines and sines instead of circular (i.e. "normal") sines and cosines.

Exercise **Name:** ____________________________

This exercise will be graded as a homework problem! Please hand it in.

The "Lorentz Transformation" given by Equations 39, 40, 41, and 42 change reference frames in special relativity. In this activity, you will go through the mechanics of a Lorentz transformation *graphically*. You are encouraged to check your answers using the equations, but by working through this on a graph, you will get a better physical feel for what these transformations are really about.

Imagine that your friend David is on a new space shuttle heading for α-Centauri. Your reference frame is designated by (x, t) and David's by (x', t'). Answer the following questions using the accompanying graph, which gives you a (x, t) coordinate axis. Get numerical values as best you can. The graph includes a dashed line ("c") showing $x = ct$. Also shown are the x' and t' axes, that is, the lines along which $t' = 0$ and $x' = 0$, respectively. The black dots show the points $(x', t') = (0, 0)$, $(0, 1)$, $(1, 0)$, $(0, -1)$, and $(-1, 0)$.

1. What is David's velocity in your reference frame? You will find it useful to draw the "world line" for a particle with a fixed value of x'.

2. What is your velocity in David's reference frame?

3. In David's reference frame, he measures the time between two events to be 1 sec. What do you say the time is? You can let the events be at the same place in David's frame.

4. In your reference frame, you measure the time between two events to be 1 sec. What does David say the time is? To do this, draw a line parallel to David's x' axis.

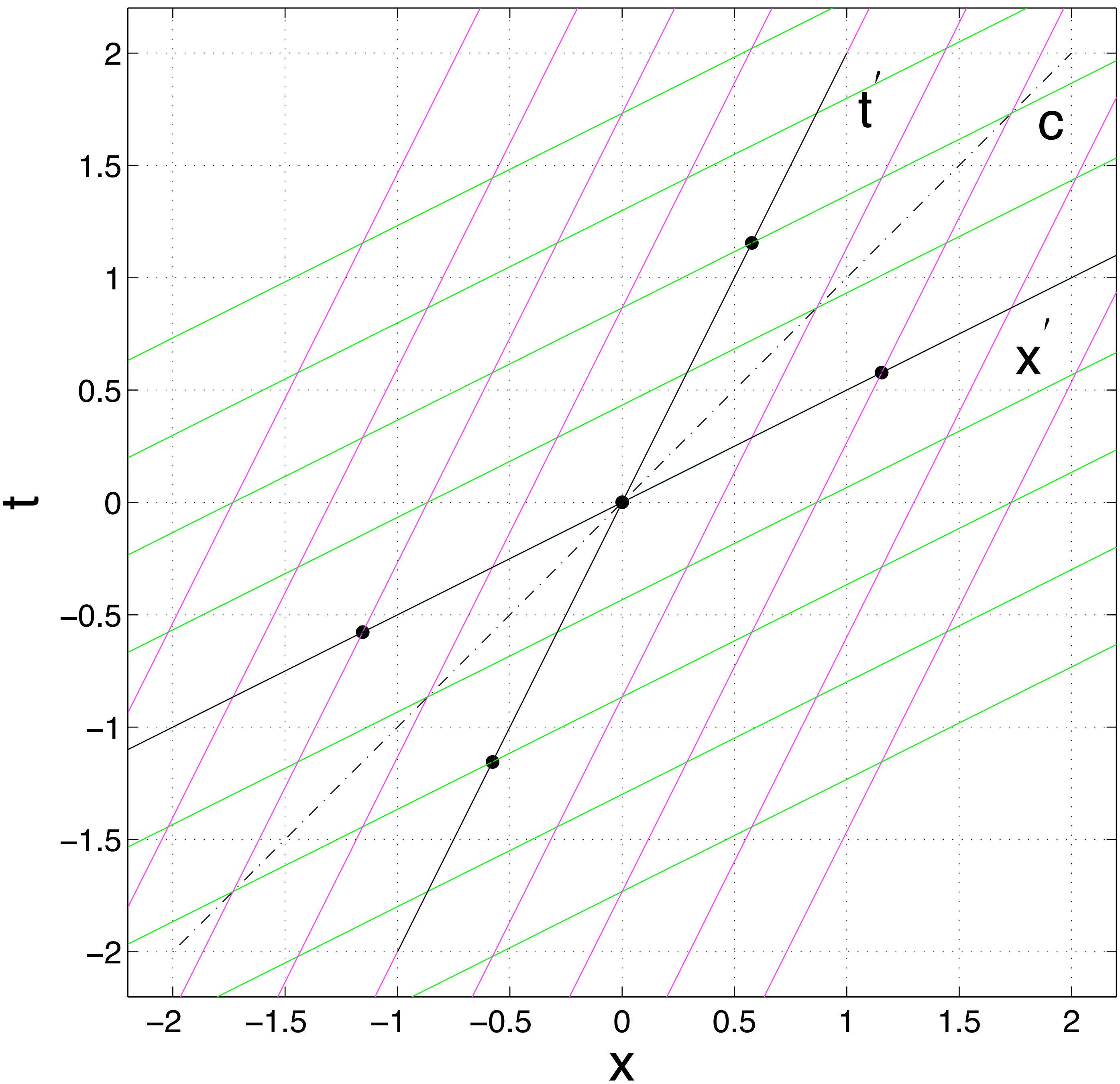
5. In your reference frame, you measure the distance between simultaneous events to be 1 light-sec. What does David say the distance is?

6. In David's reference frame, he measures the distance between simultaneous events to be 1 light-sec. What do you say the distance is? To do this, you need to draw a line parallel to David's t' axis.

7. Consider an object moving between two points. The first point is at $x = 0$ and $t = 0$. The second point is at $x = 2$ and $t = 1/2$. How fast is this object moving in your reference frame?

8. When and where are these points in David's reference frame?

9. What does this tell you about an object going faster than the speed of light?



t′
c
x′
t
x
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2