QUESTIONS AND ANSWERS

Contributions to this section, both Questions and Answers, are welcomed. Please submit four copies to the editorial office. Please include a *title* for each submission, include name and address at the end, and put references in the standard format used in the American Journal of Physics. For further suggestions, sample Questions and Answers, and requested form for both Questions and Answers, see Robert H. Romer, "Editorial: 'Questions and Answers,' a new section of the American Journal of Physics," Am. J. Phys. **62** (6) 487–489 (1994).

Questions at any level and on any appropriate AJP topic, including the "quick and curious" question, are encouraged.

Editorial Note on Answers to Question #55. Are there pictorial examples that distinguish covariant and contravariant vectors?

We reviewed over a dozen responses to Neuenschwander's question[1] that were sent in by readers. Although many of them make similar and overlapping points, we have chosen to print the following three answers as ones which cover most clearly the several aspects of the question posed. Napolitano and Lichtenstein,[2] as well as Schmidt,[3] make the point that at a conceptual level, covariant and contravariant vectors are different kinds of geometric objects, but, given a metric, there is a natural identification between them. As a result, in a space with a metric, one may speak of covariant and contravariant components of either type of vector.

While the conceptual distinction between a (contravariant) vector and a co(variant)-vector is sometimes important, because physical situations almost always involve metric spaces, and because our intuitions are so deeply encoded with notions of distances and angles, it is actually harder than one might first suppose to communicate to students why one must distinguish the two types of geometric objects. The gradient of a function (the prototype of a covector), and the flow velocity of a particle [the prototype of a (contravariant) vector] are good places to start. With the oblique, rectilinear axes in the flat plane, and with the usual metric, one may be able to provide the student some intuition in this regard. For example, if one defines a function on this plane by $f_1(x^1,x^2)=x^1$, then the lines of constant f_1 are parallel to the x^2 axis, and the gradient "vector" is perpendicular to that axis. It is easy to imagine extending this type of example to get at the more general conceptual distinction between co- and contra-variant vectors.

Neuenschwander's question also asked about pictorial illustrations. The most popular (and appropriate) response we got involved the use of oblique axes in the Euclidean plane with the usual metric. As Evans's[4] answer points out, this is a good starting example for clarifying the distinction between co- and contra-variant components. However, all the answers we got were either incomplete or too cryptic regarding the full set of circumstances where such distinctions may be usefully maintained. The metric, in the case of flat space with oblique axes, is nondiagonal. But an orthogonal, curvilinear co-ordinate system (polar co-ordinates, for example) in flat space also gives rise to a distinction between covariant and contravariant vector components. Indeed, as discussed in Mary Boas's popular text, in such cases "any vector has three kinds of components: contravariant, covariant and what we might call ordinary components."[5] Here, the metric is diagonal, but is not the identity matrix as it would be in orthogonal, rectilinear co-ordinates (aka Cartesian co-ordinates). Of course, when the space is not flat, but smoothly curved, since it is impossible to introduce Cartesian co-ordinates in a whole neighborhood, the distinction between the two kinds of vector components becomes mandatory.

[1]Dwight E. Neuenschwander, "Question #55. Are there pictorial examples that distinguish covariant and contravariant vectors?," Am. J. Phys. **65** (1), 11 (1997).

[2]J. Napolitano and R. Lichtenstein, "Answer to Question #55," Am. J. Phys. **65** (11), 1037–1038 (1997).

[3]Hans-Jürgen Schmidt, "Answer to Question #55," Am. J. Phys. **65** (11), 1038 (1997).

[4]James Evans, "Answer to Question #55," Am. J. Phys. **65** (11), 1039 (1997).

[5]Mary L. Boas, *Mathematical Methods in the Physical Sciences* (Wiley, New York, 1983), 2nd ed., Chap. 10, Sec. 13, pp. 447–449.

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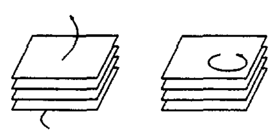
Answer to Question #55. Are there pictorial examples that distinguish covariant and contravariant vectors?

Pictorial examples as called for by Neuenschwander's question[1] indeed provide a very useful device for making the distinction between covectors (i.e., covariant vectors) and vectors (i.e., contravariant vectors), and they are used in some textbooks. For example, see the popular textbook by Misner, Thorne, and Wheeler.[2] We prefer using examples that are a bit tongue-in-cheek.

Consider first a vector. This is, in fact, the object most familiar to students, which is drawn simply as a "stick" (e.g., "→"). We refer to such an object as a "stick vector." One example is the displacement vector between two points in coordinate space. The magnitude of a stick vector is simply proportional to the length of the stick drawn on the blackboard. The arrow demonstrates the sense of direction of the stick vector.

Now consider a covector. This should be familiar to most students in terms of a gradient. We can picture a gradient best in terms of the equipotential surfaces to which it refers, and this is the basis of the pictorial representation. That is, draw the surfaces themselves, along with some sense of direction, which might be indicated by a wavy line with an arrow at the end, or with a whorl on one of the sheets:



Note that, in any case, the magnitude of the covector is proportional to the *density* of sheets.

We refer to this pictorial representation of a covector as a "lasagna vector," the sheets reminiscent of the noodles in a pan of lasagna. Students have no trouble remembering this analogy. It is also handy because if there are many noodles packed closely together in the pan, the lasagna is certainly worth more. That is, it has a larger magnitude.

Next we point out that the inner product can only be taken between a stick vector and a lasagna vector, but never between two of the same kind. The inner product is given, pictorially, by placing the stick vector into the pan of lasagna (of course, maintaining its orientation) and counting the number of noodles pierced by the stick. Clearly, the value of the inner product is both proportional to the length of the stick (i.e., the magnitude of the vector) and the density of the lasagna noodles (i.e., the magnitude of the covector).

Now a student may ask, as happened in our class, why there is any distinction between vectors and covectors since one can easily draw stick vectors for gradients by attaching them at right angles to the equipotentials, or contour lines in two dimensions. This is an excellent question and it strikes to the heart of the meaning of the metric tensor. Given our representation of vector and covector, there is not yet any way to define an angle!

An angle is defined by the inner product between two stick vectors, but as we have defined it, this operation is not possible. We need some mechanism for turning a stick vector into a lasagna vector. We would then take the inner product between this transformed stick vector and the untransformed one that remains. The object (or function) which maps a stick vector into the corresponding lasagna vector is called the metric tensor.

[1]Dwight E. Neuenschwander, "Question #55. Are there pictorial examples that distinguish covariant and contravariant vectors?," Am. J. Phys. **65** (1), 11 (1997).

[2]Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler, *Gravitation* (W. H. Freeman and Company, San Francisco, 1973), pp. 53–59, with examples throughout the book.

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Answer to Question #55. Are there pictorial examples that distinguish covariant and contravariant vectors?

Neuenschwander[1] asked how to visualize the distinction between co- and contravariant vectors. Most textbooks introduce this distinction on an abstract level; the only exception I know is that of Stephani,[2] and below I will show how I present it in my lectures "Introduction to Differential Geometry" at Potsdam University.

If *no metric exists* at all, then covariant vectors and contravariant vectors are different types of objects.

If *a metric exists*, then there is a canonical isomorphism between them; so we introduce *vectors*, and after fixing a coordinate system, we speak about their covariant and their contravariant components.

In the following, we will deal with the second case only, because it is easier to visualize: The chalkboard has a canonical metric which makes it a flat two-dimensional Riemannian manifold.

Neuenschwander[1] wrote that the mentioned distinction is necessary when dealing with curved spaces. This is not wrong, but it is a little bit misleading, and I prefer to say: ". . . is necessary when dealing with a non-Cartesian coordinate system." Example: We fix a point (the "origin" O) in the Euclidean plane; then there is a one-to-one correspondence between points and vectors. (The point P is related to the vector $\overline{OP}$.) First, we use rectangular coordinates. We might call them x and y; however, as we are interested in seeing how the situation is changed by introducing nonrectangular coordinates, we call them x^i with $i \in \{1,2\}$. So the point P has coordinates (x^1, x^2); cf. diagram 1.

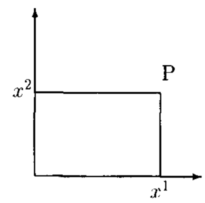

diagram 1

The coordinate system is a rectangular one, and so the component x^1 can be equivalently described as the perpendicular projection to the x^1 axis or as the projection parallel to the x^2 axis.

Let us now consider the case of an inclined system (see diagram 2). Let the angle between the axes be α with

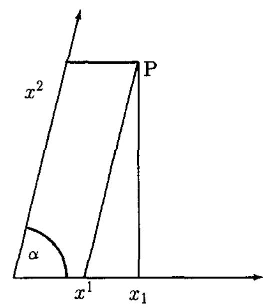

diagram 2

$0 < \alpha < \pi$.

Here, x^1 is the projection parallel to the x^2 axis, and x_1 is the perpendicular projection to the x^1 axis. We get $x_1 = x^1 + x^2 \cos \alpha$, i.e., $x_1 = x^1$ if and only if $\alpha = \pi/2$. In general, we get the following linear relation:

$$x_i = g_{ij} x^j$$

by the use of the metric g_{ij}, where $g_{12} = g_{21} = \cos \alpha$, $g_{11} = g_{22} = 1$, and summation over $j \in \{1,2\}$ is automatically assumed.

[1]D. Neuenschwander, Am. J. Phys. **65** (1), 11 (1997).

[2]H. Stephani, *General Relativity* (Cambridge U. P., Cambridge, England, 1990), 2nd ed., p. 26. (In the first German edition, which appeared in Berlin in 1997, this distinction is on p. 35.)

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