

Electromagnetic Units

Two divergent systems of units established themselves over the course of the 20th century. One system, known as *SI* (from the French *Le Système International d'Unités*), is rooted in the laboratory. It gained favor in the engineering community and forms the basis for most undergraduate curricula. The other system, called *Gaussian*, is aesthetically cleaner and is much favored in the theoretical physics community. We use *Gaussian* units in this book, as do most graduate level physics texts.

The *SI* system is also known as *MKS* (for meter, kilogram, second), and the *Gaussian* is called *CGS* (for centimeter, gram, second) units. For problems in mechanics, the difference is trivial, amounting only to some powers of ten. The difficulty comes when incorporating electromagnetism, however, where charge, for example, actually has different dimensions for the two sets of units.

This appendix attempts to contrast the two systems of units, with respect to electromagnetism. Some formulas are given which should make it easy for the reader to follow the discussions in this and other graduate level textbooks.

A.1. COULOMB'S LAW, CHARGE, AND CURRENT

Coulomb's law is the empirical observation that two charges Q_1 and Q_2 attract or repel each other with a force F_Q that is proportional to the product of the charges and inversely proportional to the square of the distance r between them. It is most natural to write this as

$$F_Q = \frac{Q_1 Q_2}{r^2} \quad \text{Gaussian} \quad (\text{A.1.1})$$

This is in fact the starting point for defining *Gaussian* units. The units of charge are called *statcoulombs*, and the force between two charges of one statcoulomb each separated by one centimeter is one dyne.

It is easy to see why such a delightfully simple formulation caught on in the physics community. Unfortunately, though, it is difficult to realize experimentally. It is much easier to set up a current source in the laboratory, perhaps with a battery driving a circuit with an adjustable resistance. Furthermore, magnetic forces between long wires are straightforward to measure. Therefore, the *SI* system is borne out of the definition of the *ampere*, namely

One ampere is that steady current which, when present in each of two long parallel conductors, separated by a distance d of one meter, results in a force per meter of length F_l/L between them numerically equal to 2×10^{-7} N/m.

The simple force formula for the *SI* system, analogous to Coulomb's law for the *Gaussian* system, is

$$\frac{F_l}{L} = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d} \quad \text{SI} \quad (\text{A.1.2})$$

for currents I_1 and I_2 (measured in amperes) in each of two wires. Although (A.1.2) doesn't carry a popularized name, it is as fundamental to the *SI* system of units as Coulomb's law (A.1.1) is to the *Gaussian* system.

TABLE A.1: Maxwell's equations in the absence of media.

| | <i>Gaussian</i> units | <i>SI</i> units |
|-------------------|---|---|
| Gauss' Law (E) | $\nabla \cdot \mathbf{E} = 4\pi\rho(\mathbf{x})$ | $\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho(\mathbf{x})$ |
| Gauss' Law (M) | $\nabla \cdot \mathbf{B} = 0$ | $\nabla \cdot \mathbf{B} = 0$ |
| Ampere's Law | $\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{c \partial t} = \frac{4\pi}{c} \mathbf{J}$ | $\nabla \times \mathbf{B} - (\epsilon_0 \mu_0) \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$ |
| Faraday's Law | $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$ | $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$ |
| Lorentz Force Law | $\mathbf{F} = Q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$ | $\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ |

Based on the definition of the ampere, we must have

$$\mu_0 \equiv 4\pi \times 10^{-7} \text{ N/A}^2 \quad (\text{A.1.3})$$

Factor of 4π frequently appear in formulations of electromagnetism because one is always bound to integrate over the unit sphere. It is a matter of taste, and now convention, whether to take them out in the beginning or carry them around at the end.

If one defines a unit of charge called the *Coulomb* as the charge passing through a wire carrying a current of one ampere during a time of one second, then Coulomb's law becomes

$$F_Q = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{r^2} \quad \text{SI} \quad (\text{A.1.4})$$

With this definition of the proportionality constant, one eventually shows that the speed of electromagnetic waves in free space is

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \quad (\text{A.1.5})$$

In our current standard units, the speed of light c is a defined quantity. Hence, ϵ_0 is also defined to be an exact value.

A relation like (A.1.5) is of course no surprise. Electric and magnetic fields are related to each other through Lorentz transformations, so the proportionality constants ϵ_0 and μ_0 should be related through c . In *Gaussian* units, there are no analogues of ϵ_0 or μ_0 , but c appears explicitly instead.

A.2. CONVERTING BETWEEN SYSTEMS

Electromagnetism can be developed by starting with (A.1.1) or (A.1.4) and incorporating special relativity. For example, one first writes down Gauss' Law as

$$\nabla \cdot \mathbf{E} = \rho(\mathbf{x})/\epsilon_0 \quad \text{SI} \quad (\text{A.2.1a})$$

$$\text{or} \quad \nabla \cdot \mathbf{E} = 4\pi\rho(\mathbf{x}) \quad \text{Gaussian} \quad (\text{A.2.1b})$$

for the electric field $\mathbf{E}(\mathbf{x})$. The remaining Maxwell's equations are then determined. Table A.1 displays Maxwell's equations in the two sets of units, as well as the Lorentz force law, in vacuum. From here, all else follows, and one can derive all the results in electromagnetism using one set of units or another.

Of course it is easiest to take one set of derivations and convert into the other after the fact. For example, (A.1.1) and (A.1.4) tell us that to make the conversion

$$\text{Gaussian} \rightarrow \text{SI} \quad (\text{A.2.2})$$

for Gauss' law, we just make the change

$$Q \rightarrow \frac{1}{\sqrt{4\pi\epsilon_0}} Q \quad (\text{A.2.3})$$

Then, referring to the Lorentz force law in Table A.1 we see that

$$\mathbf{E} \rightarrow \sqrt{4\pi\epsilon_0} \mathbf{E} \quad (\text{A.2.4})$$

$$\text{and} \quad \mathbf{B} \rightarrow c\sqrt{4\pi\epsilon_0} \mathbf{B} = \sqrt{\frac{4\pi}{\mu_0}} \mathbf{B} \quad (\text{A.2.5})$$

If you are ever confused, always try to relate things back to a purely mechanical quantity like force or energy. For example, the potential energy for a magnetic moment in a magnetic field is

$$U = -\boldsymbol{\mu} \cdot \mathbf{B} \quad (\text{A.2.6})$$

independent of which system of units we are using. Therefore, using (A.2.5) we have

$$\boldsymbol{\mu} \rightarrow \sqrt{\frac{\mu_0}{4\pi}} \boldsymbol{\mu} \quad (\text{A.2.7})$$

and so, referring to the starting point of this textbook, the magnetic moment of a circulating charge Q with angular momentum \mathbf{L} is

$$\begin{aligned} \boldsymbol{\mu} &= \frac{Q}{2mc} \mathbf{L} && \text{Gaussian} \\ \rightarrow \sqrt{\frac{\mu_0}{4\pi}} \boldsymbol{\mu} &= \frac{Q}{\sqrt{4\pi\epsilon_0}} \frac{1}{2mc} \mathbf{L} && (\text{A.2.8}) \end{aligned}$$

$$\text{or } \boldsymbol{\mu} = \frac{Q}{2m} \mathbf{L} \quad \text{SI} \quad (\text{A.2.9})$$

It is also useful to keep in mind that quantities like Q^2 have dimensions of energy \times length in *Gaussian* units. This is generally enough so that you never have to worry about what a “statcoulomb” really is.