FIGURE 6.3: Scattering through angle θ, where $\mathbf{q} = \mathbf{k} - \mathbf{k}'$.

after inserting a complete set of states $|\mathbf{x}'\rangle$ into Eq. 6.2.13. In other words, apart from an overall factor, the first-order amplitude is just the three-dimensional Fourier transform of the potential V with respect to $\mathbf{q} \equiv \mathbf{k} - \mathbf{k}'$.

An important special case is when V is a spherically symmetric potential. This implies that $f^{(1)}(\mathbf{k}', \mathbf{k})$ is a function of $q \equiv |\mathbf{q} \equiv \mathbf{k} - \mathbf{k}'|$, which is simply related to kinematic variables easily accessible by experiment. See Figure 6.3. Since $|\mathbf{k}'| = k$ by energy conservation, we have

$$q = |\mathbf{k} - \mathbf{k}'| = 2\,k\,\sin\frac{\theta}{2} \tag{6.3.4}$$

We can perform the angular integration in Eq. 6.3.3 explicitly to obtain

$$\begin{aligned} f^{(1)}(\theta) &= -\frac{1}{2}\frac{2m}{\hbar^2}\frac{1}{iq}\int_0^\infty \frac{r^2}{r} V(r)(e^{iqr} - e^{-iqr})\,dr \\ &= -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty r V(r)\sin\,qr\,dr. \end{aligned} \tag{6.3.5}$$

A simple but important example is scattering by a finite square well, that is

$$V(r) = \begin{cases} V_0 & r \le a \\ 0 & r > a \end{cases} \tag{6.3.6}$$

The integral in Eq. 6.3.5 is readily done, and yields

$$f^{(1)}(\theta) = -\frac{2m}{\hbar^2}\frac{V_0 a^3}{(qa)^2}\left[\frac{\sin qa}{qa} - \cos qa\right] \tag{6.3.7}$$

This function has zeros at $qa = 4.49, 7.73, 10.9\ldots$ and the position of these zeros, along with Eq. 6.3.4, can be used to determine the well radius a. Figure 6.4 shows elastic proton scattering from several nuclei, each of which are isotopes of calcium. The nuclear potential is approximated rather nicely by a finite square well, and the differential cross section shows the characteristic minima predicted by Eq. 6.3.7. Furthermore, the data indicate that as neutrons are added to the calcium nucleus, the minima appear at smaller angles, showing that the nuclear radius in fact increases.

Another important example is scattering by a Yukawa potential

$$V(r) = \frac{V_0 e^{-\mu r}}{\mu r} \tag{6.3.8}$$

where V_0 is independent of r, and $1/\mu$ corresponds, in a certain sense, to the range of the potential. Notice that V goes to zero very rapidly for $r \gg 1/\mu$. For this

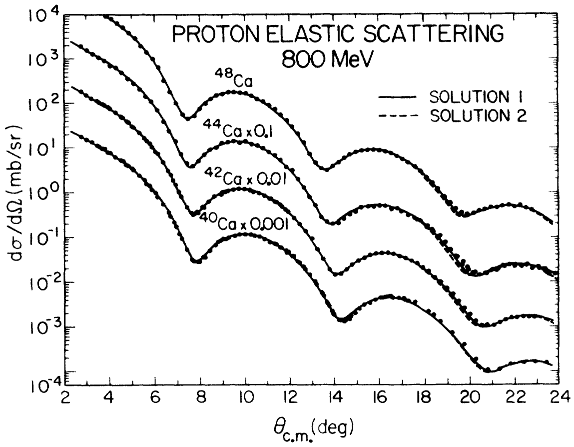

FIGURE 6.4: Data on elastic scattering of protons from the nuclei of four different isotopes of calcium. The angles at which the cross sections show minima, decrease consistently with increasing neutron number. Therefore, the radius of the calcium nucleus increases as more neutrons are added, as one expects. From L. Ray, *et al.*, Phys.Rev. C(1981)828.

potential we obtain [from Eq. 6.3.5]

$$f^{(1)}(\theta) = -\left(\frac{2mV_0}{\mu\hbar^2}\right)\frac{1}{q^2+\mu^2}, \qquad (6.3.9)$$

where we note that $\sin qr = \mathrm{Im}(e^{iqr})$ and have used

$$\mathrm{Im}\left[\int_0^\infty e^{-\mu r}e^{iqr}dr\right] = -\mathrm{Im}\left(\frac{1}{-\mu+iq}\right) = \frac{q}{\mu^2+q^2}. \qquad (6.3.10)$$

Notice also that

$$q^2 = 4k^2\sin^2\frac{\theta}{2} = 2k^2(1-\cos\theta). \qquad (6.3.11)$$

So, in the first Born approximation, the differential cross section for scattering by a Yukawa potential is given by

$$\left(\frac{d\sigma}{d\Omega}\right) \simeq \left(\frac{2mV_0}{\mu\hbar^2}\right)^2 \frac{1}{\left[2k^2(1-\cos\theta)+\mu^2\right]^2}. \qquad (6.3.12)$$

It is amusing to observe here that as $\mu \to 0$, the Yukawa potential is reduced to the Coulomb potential, provided the ratio V_0/μ is fixed—for example, to be $ZZ'e^2$—in the limiting process. We see that the first Born differential cross section