PHYS 2100 Midterm Exam #3
Introduction to Methods of Theoretical Physics
Fall 1998 NAME: ____**Solution Key**__

You have two hours to complete this exam. There are a total of *three* problems and you are to solve all of them. *Not all the problems are worth the same number of points.*

You may use your textbooks and class notes and handouts, or other books. You *may not* share these resources with another student during the test.

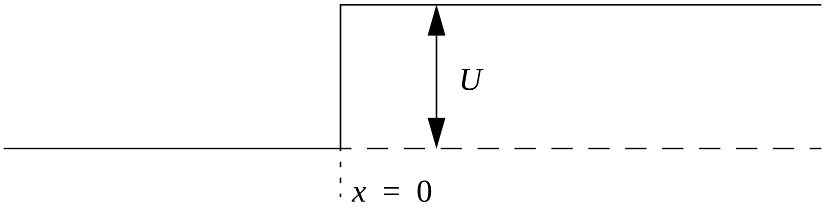
In general, if you can't get the one part of a problem, you can still do at least some of the remaining parts. In other words, *don't give up on a problem too quickly!*

Indicate any figures or tables you use in your calculations. Show all work!

GOOD LUCK!

Problem	Score	Worth
1.	________	40
2.	________	35
3.	________	25
Total Score:	___________	100

Problem 1 (5+5+5+10+15=40 Points): A particle with energy E enters from the left and encounters a potential energy step of height U located at $x = 0$:



a) Write the form of the (time-independent) wave function is the region $x < 0$. If you use any variable that is not already defined here (like k, for example) explicitly show how it is defined. You do not have to normalize your answer, but clearly show any normalization constant(s) you use.

$u(x) = Ae^{ikx} + Re^{-ikx}$ for incident and reflected waves, where $E = \frac{\hbar^2 k^2}{2m}$.

b) Assume that $E < U$. Write the wave function for $x > 0$, explicitly defining any new variables.

$u(x) = Be^{-\alpha x}$ for the wave function inside the step, where $U - E = \frac{\hbar^2 \alpha^2}{2m}$.

c) Determine the equations that satisfy the boundary conditions at $x = 0$ for the wave functions in parts **(a)** and **(b)**.

Equating functions gives $A + R = B$ and derivatives gives $ik(A - R) = -\alpha B$.

d) Find the fraction of the incident flux reflected back from the step. Explain briefly why your answer is, or is not, what you would expect from classical physics.

The two equations in **(c)** imply that $ik(A - R) = -\alpha(A + R)$ which is easily solved for $\frac{R}{A} = \frac{\alpha + ik}{\alpha - ik}$. The fraction reflected is $\left|\frac{R}{A}\right|^2 = \frac{R}{A}\left(\frac{R}{A}\right)^* = \frac{\alpha + ik}{\alpha - ik}\left(\frac{\alpha - ik}{\alpha + ik}\right) = 1$. This is just what you expect; the particle should always come back.

e) Repeat **(b,c,d)** for the assumption that $E > U$. *(There is more paper on the next page.)* Take the limit $E \to \infty$ to see if you get the answer that you expect from classical physics.

For $E > U$ we have $u(x) = Te^{i\alpha x}$ where $E - U = \frac{\hbar^2 \alpha^2}{2m}$. The other steps are the same, but this time $\frac{R}{A} = \frac{\alpha - k}{\alpha + k}$ and $\left|\frac{R}{A}\right|^2 = \left(\frac{\alpha - k}{\alpha + k}\right)^2 < 1$. *Some* is reflected, but since $\alpha \to k$ as $E \to \infty$, the reflected fraction goes to zero, again as expected.

Problem 2 (5+10+5+5+5+5=35 points). A particle is bound in a potential $V(x) = \frac{1}{2}m\omega^2x^2$ for $x > 0$ but there is an infinitely high potential wall at $x = 0$.

a) Explain briefly why a time-independent wave function of the form $u(x) = Ax\exp(-\alpha x^2)$ (where A is a constant, and $\alpha > 0$) satisfies the boundary conditions at $x = 0$ and for $x \to \infty$.

This form satisfies $u(0) = 0$ and also $u(x) \to 0$ as $x \to 0$, as needed.

b) Determine the value of α that is needed for $u(x)$ to be a solution to the Schrodinger equation.

The solution must satisfy $-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} + \frac{1}{2}m\omega^2x^2u = Eu$ so take the derivatives $\frac{du}{dx} = Ae^{-\alpha x^2} - 2\alpha Ax^2e^{-\alpha x^2}$ and $\frac{d^2u}{dx^2} = (4\alpha^2x^2 - 6\alpha)Axe^{-\alpha x^2}$ and substitute to get $-\frac{\hbar^2}{2m}(4\alpha^2x^2 - 6\alpha) + \left(\frac{1}{2}m\omega^2x^2 - E\right) = 0$. Set $\alpha = \frac{m\omega}{2\hbar}$ to remove x^2 term.

c) Determine the energy eigenvalue E in terms of m, ω, and $\hbar$.

After removing the x^2 term from above, you have $E = \frac{\hbar^2}{2m}6\alpha = \frac{3}{2}\hbar\omega$.

d) Show that the normalization constant $A = 2\left(\frac{8\alpha^3}{\pi}\right)^{1/4}$. *(See the next page.)*

$$1 = \int_0^\infty u^2(x)dx = A^2\int_0^\infty x^2e^{-2\alpha x^2}dx = A^2\cdot\frac{1}{4}\left(\frac{\pi}{8\alpha^3}\right)^{1/2} \quad \text{so} \quad A = 2\left(\frac{8\alpha^3}{\pi}\right)^{1/4}$$

e) Show that the expectation value $\langle x\rangle = \left(\frac{2}{\pi\alpha}\right)^{1/2}$. *(See the next page.)*

$$\langle x\rangle = \int_0^\infty u(x)xu(x)dx = A^2\int_0^\infty x^3e^{-2\alpha x^2}dx = 4\left(\frac{8\alpha^3}{\pi}\right)^{1/2}\frac{1}{8\alpha^2} = \left(\frac{2}{\pi\alpha}\right)^{1/2}$$

f) Could this functional form (perhaps with a different normalization constant) also be a solution to the harmonic oscillator potential, that is, the same $V(x)$ but without the infinite wall at $x = 0$? If not, explain why not. If so, comment on whether it might be the ground state or an excited state.

The Schrodinger Equation is the same for the normal harmonic oscillator, so this functional form certainly satisfies that equation. However, the solution here has *odd* parity, so it cannot be the ground state. In fact, this problem solves for the first excited state of the harmonic oscillator, except that the normalization coefficient (as well as the expectation value) would be different if integrating instead from $-\infty$.

Problem 3 (10+5+5+5=25 points): A system starts out at time $t = 0$ with a wave function given by $\psi(x, 0) = A[3u_1(x) - 4u_2(x)]$ where $u_1(x)$ and $u_2(x)$ are orthonormal, stationary state solutions of the time-independent Schrodinger Equation with energy eigenvalues E_1 and E_2.

a) Find the numerical value of the constant A. You can assume it is a real number.

Normalize: $1 = \int_{-\infty}^{\infty} |\psi|^2 dx = A^2\left[9\int_{-\infty}^{\infty}|u_1|^2 dx + 16\int_{-\infty}^{\infty}|u_2|^2 dx + 0\right] = 25A^2$

so $A = 1/5$.

b) Write down the time-dependent wave function $\psi(x, t)$ in terms of $u_1(x)$, $u_2(x)$, E_1, E_2, and, if necessary, any fundamental constants.

The form is just $\psi(x, t) = \frac{3}{5}u_1(x)e^{-i\frac{E_1 t}{\hbar}} - \frac{4}{5}u_2(x)e^{-i\frac{E_2 t}{\hbar}}$ including normalization.

c) Assume that $u_1(x)$ and $u_2(x)$ are real functions (that is, they have no imaginary parts). Find an expression for the probability density $P(x, t)$ and, in particular, show that the time-dependent part is given by $u_1(x)u_2(x)\cos\left[\frac{E_2 - E_1}{\hbar}t\right]$ to within some factor.

$$P(x, t) = |\psi(x, t)|^2 = \frac{9u_1^2 + 16u_2^2}{25} + \frac{24}{25}u_1 u_2 \cos\left[\frac{E_2 - E_1}{\hbar}t\right]$$

d) What is the numerical value of the angular frequency of oscillation between two atomic states separated in energy by 1 eV?

From **(c)** above, and as we did in class, the angular frequency is given by

$$\omega = \frac{E_2 - E_1}{\hbar} = \frac{E_2 - E_1}{\hbar c}c = \frac{1\text{ eV}}{200\text{ MeV fm}}3\times10^8\text{ m/sec} = 1.5\times10^{15}\text{ /sec}$$

You can of course plug in the numbers in whichever way suits you best, but I chose this because I always know the value of $\hbar c$.