

Using a Pendulum to teach experiment design and error analysis.

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Abstract. An experiment based on the use of a simple pendulum to measure the local acceleration due to gravity can be used to teach experimental design to advanced physics students. This involves the estimation of uncertainties in measurement and in the values of quantities derived from them. The pendulum is ideal for this purpose because the basic physics is apparently so well understood, but just below the surface lie several problems which illustrate important points about this important skill.

1. Introduction

There is only one thing more boring than working out ‘errors’ for an experiment, and that is marking student exercises on the subject. But graduate physicists must be able to do it properly. This is the motivation behind a pendulum based exercise run each year in the third level physics laboratory at the University of Queensland. The stated aim of the exercise is to measure the local acceleration due to gravity to an accuracy of $1:10^4$, by measuring the period of a simple pendulum. The students have often done similar experiments before, but this time they have to design the experiment themselves in terms of deciding what measurements to take and how to analyse the data to achieve the required precision. They are organised into competing teams and there is a daily post-mortem where their measurements and error estimates are publicly compared. The arguments which arise help to clarify their ideas about the statistical principles behind reducing uncertainty by combining multiple measurements. Is it better to measure ten swings one hundred times or one hundred swings ten times? A few lectures given concurrently remind them of what they should know. But having worked through this, most find their results lie several standard deviations away from the known value as a result of not having accounted for ‘circular error’ and other neglected factors. This brings into focus the importance of models in interpretation of results, and rewards those students who keep good laboratory notebooks which recorded the pendulum swing amplitude. They can make a simple correction, while the others have to go back and make more measurements. The instructor smiles.

2. The role of the practical class.

Australian universities have a strong tradition of including significant amounts of laboratory classes in the programs of science disciplines. Although somewhat eroded in recent years through shortage of resources, most Physics departments aspire to maintain one afternoon per week in first year, two in second, and two or three in third year, with exercises requiring an increasing level of independence and sophistication in report writing. As a result, students are sufficiently well prepared that projects in an honours year can be original research which is not infrequently published in international journals. This flying start to research makes it possible to turn out acceptable PhD graduates a mere three to four years later.

So what do students learn in practical classes in Physics? Of course a major goal is to directly experience some of the phenomena covered in lectures. This raises the question of whether it is better to meet the experiment before the lecture or vice versa. Students normally prefer the latter, but the prohibitive expense of providing multiple copies of apparatus normally means that students have to cycle through a set of experiments in more or less random order. I always tell those that complain that doing an experiment before the lecture is just what a researcher must do: by definition in that case you don't know for sure what will happen, and you need to develop appropriate survival skills of skepticism and thoroughness.

Many students who are going on to become theoretical or mathematical physicists see the concrete illustration of theoretical concepts as the main benefit of practical classes and rapidly tire of the apparent perversity and unpredictability of the real world. Most do still appreciate the relatively high staff-student ratios involved and the chance to hone report writing skills. For students training to become experimental physicists, the practical laboratories provide much more. Even the most experimental of physicists is still expected to have considerable mathematical skills and a good grasp of the basic theoretical concepts. However, he or she also needs to have an appreciation of the capabilities and limitations of measuring instruments, and of how to organize one's time and energy to obtain the desired information. Although Physics prides itself on repeatable experiments with definite outcomes, a working knowledge of statistical techniques is also important, especially when dealing with stochastic processes such as nuclear decay and photon detection. But in many ways, the most important skill for an experimental physicist is the ability to put bounds of uncertainty on the values of the quantities being measured. These uncertainty bounds or 'errors' allow judgments to be made between competing theories, which often predict only small differences in some measurable quantity eg the shift in the apparent position of a star near the Sun in the case of Einstein's General Theory of Relativity, or in the Lamb shift in the frequency of a hydrogen spectral line which confirms Quantum Electrodynamics. Only if the uncertainty in the measurement is less than the difference between two competing theoretical predictions can the measurement be held to support one rather than the other. Such considerations are not ignored in other hard sciences, but in Physics they rule.

3. Propagation of uncertainty and experiment design

From their first laboratory class in first year (and in high school in traditional syllabi) Physics students are taught to estimate uncertainties in their measurements. But one rarely measures a desired quantity directly: for example, to measure the acceleration due to gravity g by measuring the increase in velocity of a falling object in unit time is rarely convenient, and measuring the period of a pendulum and its length allow the value of g to be obtained indirectly. The process of estimating the uncertainty in the value of a desired quantity like g from the uncertainties in measurables like period and length is called 'propagation of uncertainty' and generally forms the most boring and rushed part of any practical session. Without the worry of a career destroyed by reaching a wrong conclusion, it is difficult to motivate students to take the process seriously, especially as it is so labour intensive for a supervisor to check the calculations. But with the labours of sufficiently pedantic and fastidious staff, students are eventually trained to do the job.

This happens throughout their first and second year practical laboratory experience, and continues in third year.

In their first and second year practical sessions, most of the experiments are set pieces with the apparatus provided and perhaps even connected up to save time. In third year, with more time and more staff help available in smaller classes, students are generally 'thrown in at the deep end' at least some of the time as they are ready to learn how to organize their own experiments. The next step is to learn how to design an experiment starting with an empty table.

An important factor in experiment design is to carry out a virtual uncertainty propagation analysis to determine what sort of measurements need to be made in order to obtain a final result of sufficient accuracy for the purpose required. The uncertainty in the final results will be more dependent on the uncertainty of those factors which most strongly influence the value, and effort put into reducing the uncertainties of the other factors is largely wasted. For a number of years we have run a structured exercise in our third year Physics laboratory to provide training in these issues. The actual experiment is based on the simple pendulum. The goal of the exercise is to measure the acceleration due to gravity, g , with an accuracy approaching 1 part in 10^4 .

4. The role of the pendulum in tertiary physics studies

Third year physics students are very familiar with certain aspects of pendulum behaviour. First year dynamics courses always include simple pendulum motion but the analysis is normally limited to the small angle approximation and the associated isochronism. Most students will have met the 'physical pendulum' and concepts of moment of inertia and the relationship between torque and angular momentum, but Newtonian mechanics is less deeply treated than in previous times, mainly due to the downward pressure of more advanced and modern topics from higher years, like relativity and quantum physics. Terms like 'Circular error' or 'Foucault's pendulum' generally mean little to them. However, the basic pendulum, rebadged as the simple harmonic oscillator, is ubiquitous, in Lagrangian and Hamiltonian dynamics, in quantum mechanics and quantum electrodynamics, in molecular theory, and condensed matter physics. Some extreme effects of nonlinearity are treated in courses on chaos. Indeed, variants of the pendulum are still active research areas. (eg Hensinger et al.2001) Basing our experiment on the pendulum therefore allows the students to start from a relatively well known base.

5. The 'Gravity Exercise'.

Our exercise is normally carried out over four three hour sessions. Before each session the students are asked to carry out some preparatory work which is handed in and marked to ensure it is done.

For the first session the students are asked to list five different ways to measure g , and to write out a derivation for the period of a simple pendulum, which is marked on style, with marks deducted for failing to define symbols, not stating assumptions and so on. It is hoped that such feedback at this point leads to better reports later in the semester.

The desired equation is

$$T = 2\pi\sqrt{l/g} \quad (1)$$

where T is the period, and l is the pendulum length.

Ways of measuring g range from the predictable falling body, through inclined planes, sphere rolling on concave surface, Atwood's machine....Most seem to come via Google, and Kater's pendulum (Worsnop and Flint 1957) rarely turns up. Although this would probably be the best way to derive a reliable value of g without high technology equipment, it would not provide the 'learning the hard way' experience that the simple pendulum turns out to provide. Oblivious of this, class discussion of the pros and cons of the various methods leads to general acceptance of the prospect of using a simple pendulum for our measurement. (As a complement, until this year it has been possible to make an excursion at the end of the exercise to the CSIRO Exploration Geophysics laboratory at Pinjarra Hills to see an FG5 gravity meter. This device measures the acceleration of a falling corner cube reflector in vacuum, using a laser interferometer to measure the position and a cesium clock to measure the time, and represents the state of the art in routine measurement of g .)

The rest of the session is given over to a revision lecture on propagation of uncertainties to derive the general relation for the uncertainty σ_f in some quantity which is a function $f(x,y)$ of measurands x,y with uncertainties σ_x and σ_y respectively

$$\sigma_f^2 = (\partial f/\partial x)^2 \sigma_x^2 + (\partial f/\partial y)^2 \sigma_y^2 \quad (2)$$

As preparation for the next session the students are asked to apply this equation to the case of deriving the value of g from measurements of the length and period of a pendulum in order to estimate the accuracy with which the measurements need to be made to achieve a result for g accurate to $1:10^4$.

They discover that

$$(\sigma_g/g)^2 = (\sigma_l/l)^2 + 4(\sigma_T/T)^2 \quad (3)$$

which means that the pendulum length measurement needs to be accurate to better than $1:10^4$ and the period measurement better again. Although the final uncertainty is more sensitive to the time measurement, time is generally one of the easiest quantities to measure accurately. To go further requires assumptions about the actual dimensions and means of measurement. The different options are discussed in class and it is agreed that the best we can do for the length l is to make the length as long as can be measured with a cathetometer which can measure up to a metre with a claimed uncertainty of less than 0.05mm. The students are told that the period will have to be measured using a stopwatch and led into a discussion and a lecture on sampling statistics and the relationship of the standard deviation of the mean of n measurements to the standard deviation of the parent population, which gives a factor \sqrt{n} reduction in uncertainty when n repeated measurements are averaged. A discussion of timing using a stopwatch leads to the

conclusion that the reaction time delays at each end can largely cancel leaving the random variation in reaction time to dominate the uncertainty. But since this is independent of the number of periods counted, the uncertainty in the period can be made small by counting a large number (m) of swings and dividing the total time by the number. If this process is repeated n independent times, a net uncertainty reduced by the factor $m\sqrt{n}$ relative to measurement of a single swing can result. Given a fixed maximum time available for measurement, this factor is maximized by making m as large as possible, but it is desirable to repeat the experiment at least a few times so as to generate an estimate of σ_T . Based on guesses of this uncertainty, and that of l , the conclusion is reached that although an accuracy of $1:10^4$ is not achievable, a result better than $1:10^3$ should be.

At this point, a short lecture is given on the now standard ISO methodology for the treatment of uncertainties in measurement. (ISO, 1993). This methodology explicitly distinguishes between uncertainties derived from analysis of repeated measurements ('type A'), and those estimated from specifications or otherwise ('type B') and provides guidelines for reporting of uncertainties.

At the next session the students are divided into three teams and each team is presented with a pendulum, a stopwatch, and shown how to use the cathetometer. The pendulum consists of a brass ball of 50mm diameter attached to a 950mm length of steel piano wire 0.5mm diameter which is in turn attached to a hardened steel ring (actually the outer race of a roller bearing). The steel ring can be hung on a knife edge consisting of length of square tool steel rigidly attached to a concrete block wall by a massive steel bracket. The teams are told to measure extra parameters like the dimensions of the pivot ring and the wire thickness for future reference but are deliberately not reminded of the small angle approximation underlying their analysis. Toward the end of the session the period of each pendulum is measured for very small oscillations using a Microset electronic timer (Mumford Microsystems, Santa Barbara CA) by one of the supervisors.

Before the next session, the students have to carry out a computational exercise using Matlab to illustrate the results of the Central Limit Theorem. Starting with 1000 random numbers uniformly distributed over an interval, they show that if the numbers are broken up into 100 groups of ten and the average is taken of each group, then the averages have a more or less normal distribution with a standard deviation smaller than that of the parent population by the factor $\sqrt{10}$. At the session, after a lecture on using weighted means to combine separate measurements of the same quantity, the three teams compare their measurements, and more importantly, their uncertainty estimates. Different teams have usually adopted somewhat different measurement strategies so some interesting discussions arise. Most groups find that the (now type A) uncertainty in period reflecting the variation in their repeated measurements, is in reasonable agreement with that predicted in class (type B). This is reassuring, showing that an experiment can be designed, rather than just attempted.

However, it normally emerges that all of the period estimates are significantly, ie several standard deviations, higher than the measurements made with the Microset timer, and when values of g are calculated these lie significantly above the accepted value. (eg $9.820 \pm 0.003 \text{ ms}^{-2}$ rather than $9.7915330 \pm 1 \text{ ms}^{-2}$.) What is more, the values obtained

by the three teams all differ by amounts larger than their calculated uncertainties. Of course, if the uncertainty estimates have any value, this should not happen.

At this stage it seems that the whole system has broken down, but the paradox is easily resolved with the clue given by the Microset timer, and students soon see that the circular error even for an amplitude of 100mm is significant. Surprisingly (or is it?) the different teams usually carefully control the initial oscillation amplitude of each of their own measurements to some more or less arbitrary value, so that the circular error becomes an unrecognized systematic error, rather than causing a spread in the results which would be reflected in an increased uncertainty. In many cases, the teams have dutifully recorded the amplitude and are in a position to make a correction. The others are punished by having to carry out some more measurements. But even using the Microset timer results taken with small amplitudes, the value of g obtained does not agree with the accepted local value of g within the calculated uncertainty.

6. Corrections to the simple theory

The rest of the exercise relies heavily on a paper by Nelson and Olsson (Nelson and Olsson 1986) which describes how to make a series of corrections to the theoretical model to take into account the factors ignored in the simple pendulum model. Apart from the ‘circular error’ correction, these include the finite size and moment of inertia of the bob and support ring and wire, and buoyancy and damping due to air.

Since each of these corrections is small, we can estimate them separately, and then simply add them. It is convenient to do this by correcting the value of the period to derive a value for the equivalent simple pendulum of the same length, so that the value of g can be derived from the standard simple equation. Thus, assuming that the result of a more complete theory accounting for one of these effects produced an expression for the period T , we can write it in the form

$$T = T_0 + \Delta T = T_0 (1 + \Delta T / T_0) \quad (4)$$

where T is the period calculated by the more complete theory, and T_0 is that for the equivalent simple pendulum. This can be inverted to give

$$T_0 = T (1 - \Delta T / T) + \text{higher order terms} \quad (5)$$

Where T is now the measured period and ΔT is the theoretical correction. We can then use T_0 instead of T in our calculation of g . Although this procedure may seem a bit indirect it is often used in Physics as it allows the theory to be developed in a stepwise way and transparently allows the effect of each correction to be debated and settled.

6.1 FINITE AMPLITUDE (‘CIRCULAR ERROR’) CORRECTION

A convenient perturbation expansion is quoted by Nelson and Olsson:

$$\Delta T / T_0 = \theta_0^2 / 16 + 11 \theta_0^4 / 3072 + \dots \quad (6)$$

For an amplitude of order 10cm giving $\theta_0 = 0.1$, this term is at the 10^{-3} level and an important correction. Students usually seem to like such a healthy amplitude for their measurements with the stopwatch. But with the amplitude of 0.01 used with the Microset timer, the correction is below the 10^{-5} level.

6.2 MASS DISTRIBUTION CORRECTIONS

Considering a pendulum of moment of inertia I about its pivot and total mass M with centre of mass a distance h from the pivot, the period for small oscillations is

$$T = 2\pi\sqrt{(I/Mgh)} \quad (7)$$

For a simple pendulum, $I = Mh^2$, $h = l$, and equation (1) is recovered. Otherwise, using the parallel axes theorem, corrections can be calculated for the effects of the finite size of the bob, the moment of inertia of the wire and the moment of inertia of the pivot ring. Proofs of these results are set as exercises for the students.

For the finite size of the bob, $\Delta T/ T_0 = (a/l)^2/5$ (8)

where a is the radius of the spherical bob and l is the length of the pendulum.

For the mass of the wire, $\Delta T/ T_0 = - m_w/12M$ (9)
 where m_w is the mass of the wire and M is the total mass of the pendulum.

For the metal ring on which the pendulum pivots,

$$\Delta T/ T_0 = - m_r d_r/2ml + I_r/2ml^2 \quad (10)$$

where m_r is the mass of the ring, and d_r is the distance from the centre of the ring to the pivot ie the inner radius of the ring. Values of these factors are given in Table 1 where it can be seen that the effect of the ring is quite significant.

6.3 AIR CORRECTIONS

The apparent weight of the bob is reduced by the weight of air it displaces m_a , decreasing the effect of gravity, giving a correction

$$\Delta T/ T_0 = m_a/2m \quad (11)$$

Also, as noted by Bessel, air moving with the bob adds to its effective inertia, leading to a correction

$$\Delta T/ T_0 = km_a/2m$$

where k is a constant believed to be about 1.18 for a sphere.

Air is also responsible for damping the motion of the pendulum. Although the natural frequency of a damped system is lower than that of an undamped one, this effect should not be significant here. However, if the amplitude decays from its initial value to a lower one during the time of measurement, the appropriate value of the finite amplitude correction (6) will vary. Since the decay of amplitude will not be exponential due to turbulent drag on the bob, the variation is hard to quantify. The students are asked to estimate the variation in θ_0 over their measurement period and use an average value. In their paper, Nelson and Olsson consider all of these factors in detail together with a number of others which turn out not to be important in our case. Some typical values are given in table 1.

item	Value (s)	Uncertainty (s)
Measured period	2.008 5s	0.000 2
Finite amplitude correction ($\theta_0 = 0.1$)	0.001 25	negligible
Finite bob correction	0.000 2	“
Pivot ring correction	-0.002 2	“
Wire mass correction	-0.000 6	“
Buoyancy	0.000 1	“
Damping	negligible	“
Mass of air moving	0.000 1	“
Total correction	-0.001 0	“
Corrected period ($T_0 = T - \text{total correction}$)	2.009 5	0.000 2

Table 1. Corrections to pendulum period.

Taking the corrected period from Table 1 and using equation (1) with the measured distance from the pivot to the centre of the bob ($l = 1.001\ 45 \pm 0.000\ 05$ m), leads to a value of g

$$g = 9.790\ 4 \text{ with a standard uncertainty of } 0.005 \text{ ms}^{-2}$$

which agrees well with the value established in the laboratory using a Lacoste-Romberg gravity meter of $9.791\ 533\ 03 \pm 10 \text{ ms}^{-2}$.

7. Conclusions

This is the first time that most of the students have had to make choices about how an experiment will be done. The initial goal of a measurement accurate to one part in ten thousand is not achieved, but it is known from the design process before the experiments are commenced, that with the equipment available, the uncertainty will be at least a few parts in ten thousand, and that is what is achieved.. The importance of the model used to interpret or analyse the measurements is emphasized through showing how the neglect of

effects normally regarded as ‘small’ can seriously bias the result and by organizing the students into teams and publicly discussing the results at each step, more options and problems are encountered in the time available. The pendulum once again provides the basis for a valuable learning experience, this time not so much about physical laws, but more about how to be an experimental physicist.

References

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