Please turn in your homework now!

In this class we will discuss the structure of Main Sequence Stars:

• Spheres in static equilibrium
• Stellar energy sources: Nuclear Physics
• Stellar structure and stellar models
• Outstanding issues: Solar Neutrinos

Note: No class next Tuesday, but see lab for Friday
Also: See updated HW assignment (Ch.9) due in two weeks
The Main Sequence: “Normal” stars
Example: The Pleiades

See lab this Friday.
Example: The **old** cluster M67
Spheres in static equilibrium

Build a star in thin spherical shells:

\[ dV = 4\pi r^2 dr \]
\[ dM = \rho(r) dV = 4\pi r^2 \rho(r) dr \]

**Example:** Mass of a sphere with constant density

\[ M = \int_0^R dM = 4\pi \rho \int_0^R r^2 dr = \frac{4}{3} \pi R^3 \rho \]
Gravitational potential energy

Bring two masses together from infinity

$U = 0$

$U = -\frac{Gm_1m_2}{r}$

“Negative work” is done in order to bring the two masses together. In other words, energy is made available thanks to the “gravitational collapse”.
Application: The gravitating sphere

Assemble the sphere in shells. How much energy becomes available from gravitational collapse?

\[ m_1 = M(r) \quad \text{and} \quad m_2 = dM \quad \text{so} \quad dU = -G \frac{M(r) dM}{r} \]

Make assumption of constant density:

\[ dU = -G \left[ \frac{4}{3} \pi r^3 \rho \right] [4\pi r^2 \rho dr] \frac{1}{r} = -3G \left( \frac{4\pi}{3} \rho \right)^2 r^4 dr \]

\[ U = \int_0^R dU = -\frac{3}{5} G \left( \frac{4\pi}{3} \rho \right)^2 R^5 = -\frac{3GM^2}{5R} \]
Example: How long would the Sun live if its energy was from gravitational collapse?

Answer: Estimate available energy from gravitation and use the known luminosity to get a time

\[
t = \frac{\frac{3}{5} GM^2}{R \frac{2 \times 10^{48} \text{ erg}}{L \frac{4 \times 10^{33} \text{ erg/sec}}}} = \frac{2 \times 10^{48} \text{ erg}}{4 \times 10^{33} \text{ erg/sec}} = 5 \times 10^{14} \text{ sec}
\]

\[
= 2 \times 10^7 \text{ years}
\]

This sounds like a long time, but it isn’t. We know that the solar system is over four billion years old.

i.e. The Sun gets its energy some other way!
Suppose this energy was turned into heat? How hot would this make the Sun? *(Make a rough guess.)*

Consider thermal kinetic energies of particles in the Sun:

\[
K = \frac{3}{2} nkT = \frac{3}{2} \left( \frac{M}{m} \right) kT = \frac{3}{2} \frac{1.99 \times 10^{33}}{21.67 \times 10^{-24}} \text{gm} kT \\
= 1.8 \times 10^{57} kT = 2 \times 10^{48} \text{erg}
\]

*(Assuming the Sun is made of hydrogen atoms.)*

Solve for Temperature: \( T = 0.81 \times 10^7 K \approx 8M K \)

*This is much hotter than the surface of the Sun, but the center of the Sun would be much higher pressure. This is the key to how the Sun and other stars shine!*
Nuclear Physics: Power source of stars

First the nomenclature: $^AZ$

$Z$ = “atomic number” = number of protons
$A$ = “atomic weight” = $Z+N$ = protons + neutrons

Examples:

$^1H = $ hydrogen (also “$p$”)
$^2H = $ deuterium (also “$d$”)
$^4He = $ helium (also “$\alpha$”)
$^3He = $ helium-3
$^{12}C = $ carbon-12
Hydrogen fusion: The primary reaction

We will argue that nuclear fusion is the power source behind stars. There are many possible fusion reactions. However, the most important boil down to the following:

\[ ^1H + ^1H + ^1H + ^1H \rightarrow ^4He + 2e^- + 2\nu_e \]

Energy is released, which is carried off by electrons, which heat up the star, and neutrinos which travel into space.

The different series of reactions which boil down to this result are called the “pp chain” and the “CNO cycle”. See Kutner Section 9.3 for details.
The Proton-Proton Chains

One of the two ways to burn hydrogen.

\[ p + p \rightarrow ^2\text{H} + e^+ + \nu_e \]

\[ p + e^- + p \rightarrow ^2\text{H} + \nu_e \]

“pep”

\[ p^+ + 2^1\text{H} \rightarrow ^3\text{He} + \gamma \]

\[ ^3\text{He} + ^3\text{He} \rightarrow ^4\text{He} + p + p \]

\[ ^3\text{He} + ^4\text{He} \rightarrow ^7\text{Be} + \gamma \]

\[ ^7\text{Be} + e^- \rightarrow ^7\text{Li} + \nu_e \]

\[ ^7\text{Be} + p \rightarrow ^8\text{B} + \gamma \]

\[ ^7\text{Li} + p \rightarrow ^4\text{He} + ^4\text{He} \]

\[ ^8\text{B} \rightarrow ^8\text{Be} + e^+ + \nu_e \]

\[ ^8\text{Be} \rightarrow ^4\text{He} + ^4\text{He} \]

CNO Cycle

*Carbon-12 nucleus acts as a “catalyst”.*

\[ ^{12}\text{C} + p \rightarrow ^{13}\text{N} + \gamma \]

\[ ^{15}\text{N} + p \rightarrow ^{12}\text{C} + ^4\text{He} \]

\[ ^{13}\text{N} \rightarrow ^{13}\text{C} + e^+ + \nu \]

\[ ^{15}\text{O} \rightarrow ^{15}\text{N} + e^+ + \nu \]

\[ ^{13}\text{C} + p \rightarrow ^{14}\text{N} + \gamma \]

\[ ^{14}\text{N} + p \rightarrow ^{15}\text{O} + \gamma \]
Energy from Nuclear Reactions

Stars get energy from fusion. (Nuclear reactors use fission.)

$^4\text{He}$
A Crucial Test: The lifetime of the Sun

\[4m(\text{^1H}) - m(\text{^4He}) = 0.007 \times [4m(\text{^1H})]\]

In other words, 0.7% of the proton mass is converted to energy when hydrogen fuses to form helium.

Suppose that only 10% of the hydrogen in the Sun can be used for fusion power. Then, the available energy is

\[E = 0.10 \times 0.007 \times M_\odot c^2 = 1.25 \times 10^{51} \text{ erg}\]

and the lifetime of the sun would be

\[t = E/L = 3.13 \times 10^{17} \text{ sec} \approx 10^{10} \text{ years}\]

This makes sense! The earth is about four billion years old, i.e. the Solar System is “middle aged”.
The Fusion Barrier and Temperature

Need to overcome electrical repulsion of nuclei

We rely on thermal energy to force the nuclei close enough (every so often) so that they undergo nuclear reactions at some rate.

Quantum mechanics is necessary for a full understanding!
The “Gamow Peak”

Combined effect of quantum mechanical “tunnelling” and Maxwell-Boltzmann thermal distribution.

Maxwell distribution:

\[ P \propto e^{-E/kT} \]

Tunnelling probability:

\[ P \propto e^{-ar/\lambda} \]

\[ \lambda = \frac{h}{p} = \frac{h}{mv} \quad \& \quad E = \frac{1}{2}mv^2 \]

so \[ P \propto e^{-b/E^{1/2}} \]

See Kutner Fig.9.5 also HW Prob.9.11
Stellar structure and stellar models

How do we go about “building” a (main sequence) star?

Our assumptions:

• Some elemental composition (mainly $^1H$)
• Ideal gas law (is it consistent??)
• Spherical symmetry
• Hydrostatic equilibrium
• Some model of radiation transport
Hydrostatic Equilibrium

Consider pressure and the spherical shells:

Inward force = $P(r+dr)dA$

Inward force slightly smaller than outward force because gravity also pulls inward!

Outward force = $P(r)dA$

Mass of the red slug is $dm = \rho(r)drdA$

so, the gravitational force is

\[ F_G = -G \frac{M(r)dm}{r^2} \]
“Equilibrium” means that all forces balance:

Inward forces = Outward forces

\[ P(r + dr) dA - F_G = P(r) dA \]

\[ P(r + dr) dA - P(r) dA = -G \frac{M(r) dm}{r^2} \]

\[ P(r + dr) - P(r) = -G \frac{M(r) \rho(r) dr}{r^2} \]

Now let \( dr \to 0 \):

\[ \frac{dP}{dr} = - \left[ \frac{GM(r)}{r^2} \right] \rho(r) \]

“Differential Equation” for pressure as a function of radius.
This is not the first differential equation we’ve seen!

Recall:  \[ dM = 4\pi r^2 \rho(r) \, dr \quad \text{or} \quad \frac{dM}{dr} = 4\pi r^2 \rho(r) \]

We solved this (i.e. we found \( M(r) \), and then the total mass \( M(R) \)) by making an assumption about the density, namely that it was a constant. You can also find the mass by making different assumptions. (See HW.)

We can solve for the pressure by making similar assumptions.
Example: The central pressure of a star

Assume a constant density and integrate “in one big gulp”:

\[ dr = R - 0 = R \quad \text{and} \quad dP = 0 - P_C = -P_C \]

\[ \Rightarrow \frac{-P_C}{R} = - \left[ \frac{GM}{R^2} \right] \rho = - \left[ \frac{GM}{R^2} \right] \left[ \frac{M}{4\pi R^3/3} \right] \]

So, ignoring the numerical factors for a rough estimate

\[ P_C \approx \frac{GM^2}{R^4} \]
The Missing Ingredient: The “Equation of State”

Need a relation between pressure, density, and temperature:

\[ P = f(\rho, T) \]

For main sequence stars, the Ideal Gas Law works okay:

\[ PV = nkT \quad \text{and} \quad \rho = M/V = nm/V \]

so \[ P = \left(\frac{\rho}{m}\right)kT \]

Example: The temperature at the center of a star

\[ T_c = \frac{m}{\rho k} P_c \]
Example: The Mass-Luminosity relationship

\[ L(r) = 4\pi r^2 f(r) \quad \text{where} \quad f(r) = \sigma T(r)^4 \]

Expect \( df = \kappa(r) \rho(r) f(r) dr \), \( \kappa(r) = \text{“opacity”} \)

So \( \frac{df}{dr} = \kappa \rho f(r) = 4\sigma T(r)^3 \frac{dT}{dr} \quad \Rightarrow \quad f(r) \propto \frac{T(r)^3 dT}{\rho(r) dr} \)

Therefore \( L(r) \propto \frac{r^2 T(r)^3 dT}{\rho(r) dr} \)

Now, integrate in “one big gulp” again!

\[ L = L(R) \propto \frac{R^2 T_C^3}{M/R^3} \frac{T_C}{R} = \frac{R^4}{M} T_C^4 \propto \frac{R^4}{M} \left(\frac{M}{R}\right)^4 \text{ or } L \propto M^3 \]
Recall:

A more realistic solar model:

Compare Kutner Fig. 9.11