Preliminary Examination

January 17, 1998

Ground Rules: No books, notes, or calculators are permitted. Please select 10 questions out of the following 12 questions to be graded. You have 4 hours to complete the exam.

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1. (a) By using partial fractions, evaluate the limit as $n \to \infty$ of the sequence

$$S_n = \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \ldots + \frac{1}{n(n+2)}.$$

Use the result to show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

(b) Compute the limit

$$\lim_{x \to 0} (\cos x + \sin x)^{\frac{1}{x}}.$$

2. (a) Let A, B, and C be $n \times n$ upper-triangular matrices. Let their respective diagonal elements be A_{jj}, B_{jj}, and C_{jj}, $j = 1, \ldots, n$, and let all of these elements be nonzero. Compute the determinant of the $2n \times 2n$ matrix

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix},$$

where 0 is the $n \times n$ zero matrix.

(b) Compute the determinant of the matrix

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}.$$

3. Find the Taylor series for the following functions about the point $x_0 = 0$:

$$f(x) = \frac{1}{1+x}, \qquad g(x) = \ln(1+x), \qquad h(x) = \frac{\ln(1+x)}{1+x}.$$

In each case find the largest interval of x for which the Taylor series converges absolutely.

4. (a) Find the point on the plane $x + 2y + 3z = 6$ which is closest to the origin.

(b) Find the volume of the largest rectangular box in $x > 0$, $y > 0$, $z > 0$ with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 6$.

5. Consider a lake of constant volume V containing at time t an amount $Q(t)$ of pollutant, evenly distributed throughout the lake with a concentration $c(t)$, where $c(t) = Q(t)/V$. Assume that water containing a concentration k of pollutant enters the lake at a rate r, and that water leaves the lake at the same rate. Suppose that pollutants are also added directly to the lake at a constant rate P.

(a) If at time $t = 0$ the concentration of pollutant is c_0, find an expression for the concentration $c(t)$ at any time.

(b) Is there a limiting concentration as $t \to \infty$, and if yes, what is its value?

(c) If the addition of pollutants to the lake is terminated ($k = 0$ and $P = 0$ for $t > 0$), determine the time interval before the concentration of pollutants is reduced to the fraction q of its original value.

6. Let A be the following real 3×4 matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 3 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}. \qquad \text{Let } b = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}.$$

(a) Find a basis for the null space of A.

(b) Find a basis for the range of A.

(c) Find all solutions to $Ax = b$.

7. (a) Compute the integral

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

HINT: Introduce polar coordinates into the expression for I^2.

(b) Compute the double integral

$$J = \iint_R \frac{(x-y)^2}{1+x+y}\, dx\, dy,$$

where R is the trapezoidal region bounded by the lines $x + y = 1$ and $x + y = 2$ in the first quadrant.
HINT: Use the change of variables $u = 1 + x + y$, $v = x - y$.

8. The cycloid is given parametrically by the equations

$$x = a(t - \sin t), \qquad y = a(1 - \cos t),$$

and has cusps at $t = 2n\pi$, n integer.

(a) Compute the length of the arc of the cycloid between any two successive cusps.

(b) Compute the area bounded by one arch of the cycloid (*i.e.*, the arc between two successive cusps) and the x-axis.

9. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x_1, x_2) = (-x_2, x_1)$.

(a) Show that there are no one-dimensional subspaces of $\mathbb{R}^2 =$ which are invariant under T.

(b) Show $T - cI$ is invertible for any real scalar c.

(c) Now assume $T : \mathbb{C}^2 \to \mathbb{C}^2$. Find the eigenvalues and eigenvectors of T.

10. Consider the partial differential equations

$$\frac{\partial w}{\partial t} + J(\Psi, w) = 0, \qquad \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + w = 0,$$

for two functions $w(x, y, t)$ and $\Psi(x, y, t)$, where

$$J(\Psi, w) = \frac{\partial w}{\partial x}\frac{\partial \Psi}{\partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial w}{\partial y}$$

is the Jacobian determinant of w and Ψ. Given any smooth function F of Ψ, show that Ψ and $w = F(\Psi)$ produce a t-independent solution of the above two equations, provided Ψ satisfies a certain partial differential equation. Which equation must Ψ satisfy?

11. (a) Find the values of the non-negative variables x_i, $i = 1, ..., N$ which extremize the function

$$S = \sum_{i=1}^{N} x_i \log x_i$$

under the constraint that $\sum_{i=1}^{N} x_i = X$ where X is a fixed number.

(b) Compute the following limit in terms of Y which is a positive number:

$$\lim_{n \to 0} \frac{Y^n - 1}{n}.$$

12. Compute (in any way you can) the integral

$$\oint e^x \sin y \, dx + e^x \cos y \, dy$$

around the rectangle with vertices $(0, 0)$, $(1, 0)$, $(1, \frac{1}{2}\pi)$, and $(0, \frac{1}{2}\pi)$.