Preliminary Examination
January, 2001

Ground rules: No books, notes or calculators are allowed. Please do any 10 problems. On the front page of the answer book, identify the 10 problems you wish to be graded. You have 4 hours to complete the examination.

1. If \( f(x) = \sin(ln x) \), show that \( x^2 f''(x) + xf'(x) + f(x) = 0 \).

Find the second order Taylor Polynomial approximation to \( f(x) \) at \( x = 1 \). Evaluate \( \lim_{x \to 1} \frac{f(x)}{x - 1} \).

2. (a) Do the series

\[
\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n - 1}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n \sin \left( \frac{1}{n} \right)}{n}, \quad \sum_{n=1}^{\infty} \sin^2 \left( \frac{1}{n} \right)
\]

converge? Explain.

(b) Show that the integral

\[
\int_{0}^{1} \frac{dx}{[2x - 1]^{1/2}}
\]

converges and compute its value.

3. Show that \( L : \mathbb{R}^3 \to \mathbb{R}^2 \) defined by

\[
L \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ a_3 \end{pmatrix}
\]

is a linear transformation. Is \( L \) one-to-one (or injective)? Is \( L \) onto (or surjective)? Find a basis for the null space of \( L \).

4. Find the eigenvalues and the associated eigenvectors for

\[
A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}
\]

Is \( A \) diagonalizable? If so, exhibit a matrix \( S \) and a matrix \( S^{-1} \) that diagonalize \( A \).
5. (a) Let $A$ and $B$ be given nonzero vectors in $\mathbb{R}^3$, and $\tau$ the position vector of a point. Describe the set of points determined by the equation

$$(\tau - A) \cdot (\tau - B) = 0.$$ 

(b) For a nonzero vector $A$, show whether or not it is always true that

$$A \cdot B = A \cdot C \quad \text{and} \quad A \times B = A \times C \quad \text{implies that} \quad B = C.$$ 

6. The planes $M_1$ and $M_2$ are given by the equations $x+2y+2z = 1$ and $3x-2y+6z = 3$. These planes intersect in a line $L$. Find an equation for the plane which contains $L$ and bisects the acute angle between $M_1$ and $M_2$.

7. Consider the function $f(x, y) = \sqrt{xy}$.

(a) For $(x, y)$ in the first quadrant, sketch either a graph, or some level curves, of $f$.

(b) Consider the following quantities:

i. $\nabla f$ at $(0,0)$.

ii. $Df$, the directional derivative pointing in the direction from $(0,0)$ to $(3,4)$, and evaluated at $(0,0)$.

iii. The tangent plane to the graph of $f$ at $(0,0)$.

iv. The linear approximation to $f$ at $(0,0)$.

If these quantities exist, evaluate them. If they do not, explain why not.

8. Two masses $m_1$ and $m_2$ occupy nonoverlapping regions $R_1$ and $R_2$ in $\mathbb{R}^3$. Let $P_1$ and $P_2$ be the centers of mass of $R_1$ and $R_2$ respectively. Show that the center of mass of $m_1$ and $m_2$ considered as a single mass in space is the same as it would be if the entire mass $m_1$ were concentrated at $P_1$ and the entire mass $m_2$ at $P_2$. Do not assume constant density.

Will the result hold for three masses $m_1$, $m_2$ and $m_3$ in nonoverlapping regions $R_1$, $R_2$ and $R_3$?

9. If $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, find

$$\max_x x^T A x$$

when

$$x^T B x = 1$$
10. Let $A$ be an $n \times n$ real matrix

where

$$A = A^T,$$

$$Av^k = \lambda_k v^k, \quad k = 1, 2, \ldots, n$$

and

$$(v^k)^T v^k = 1$$

and

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n.$$ 

(a) Prove

$$A = \sum_{k=1}^{n} \lambda_k v^k (v^k)^T.$$ 

(b) Find $A^{-1}$, $e^A$, and a matrix $B$ that solves $B^2 = A$ all in terms of $\lambda_k$ and $v^k$. 

11. If $\mathbf{E}(r) = \frac{r}{|r|^3}$ for $r = (x, y, z)$, a vector in $R^3$. 

Find

(a) $\text{div} \mathbf{E}(r) = \nabla \cdot \mathbf{E}(r)$ when $r \neq 0$. 

(b) $\int_S \mathbf{E}(r) \cdot \mathbf{n} \, dA$ when $S = \{(x, y, z) | (x - 2)^2 + (y - 2)^2 + (z - 2)^2 = 1\}$. 

(c) $\int_S \mathbf{E}(r) \cdot \mathbf{n} \, dA$ when $S = \{(x, y, z) | (x - 2)^2 + (y - 2)^2 + (z - 2)^2 = 100\}$. 

12. Let $C$ be a simple, smooth, closed plane curve bounding an area $A$. 

(a) Find

$$\int_C x \, dy - y \, dx$$

in terms of $A$. 

(b) If $\mathbf{E}(r)$ is a smooth vector field in $R^3$ and

$$\text{curl} \mathbf{E}(r) = \nabla \times \mathbf{E}(r) = 0$$

show there is a function $u(r)$ where $\mathbf{E}(r) = \nabla u(r) = \text{grad} u(r)$. 

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