Preliminary Examination

September 4, 1999

Ground Rules: No books, notes, or calculators are permitted. Please select 10 questions out of the following 12 questions to be graded. You have 4 hours to complete the exam.

1. (a) Does the series $\sum_{n=1}^{\infty} \sin^2\left[\pi\left(n+\frac{1}{n}\right)\right]$ converge? Explain!

   (b) Show that the integral $\int_0^{\infty} \frac{x^7\,dx}{(1+x^4)^3}$ converges, and compute its value.

2. Consider the system of equations:

$$\begin{array}{rcrcrcrcr} w & + & 2x & + & 3y & - & z & = & 1 \\ 2w & + & 4x & + & 7y & - & 2z & = & b_2 \\ -w & - & 2x & - & 4y & + & z & = & b_3 \end{array}$$

   (a) What must b_2 and b_3 satisfy for the system to be consistent?

   (b) Describe the set of solutions to the system when $b_2 = 2$ and $b_3 = -1$.

3. Find the Taylor series about the point $x_0 = 0$ for the following functions:

$$\text{(a) } \frac{1}{1+x^2}, \qquad \text{(b) } \arctan x, \qquad \text{(c) } \frac{\arctan x}{1+x^2}.$$

   In each case find the largest interval of x for which the Taylor series converges absolutely.

4. Show that if $f(x,y)$ satisfies Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

   then so does

$$\phi(x,y) = f\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right).$$

5. Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 & 1 \\ -1 & -2 & -1 & 4 & -1 \end{bmatrix}.$$

   (a) Find a basis for the null space of A.

   (b) What is the rank of A?

6. Evaluate the limits

$$\text{(a)} \lim_{x \to 0} \frac{e^{x^2} - \cos x}{x^2}, \qquad \text{(b)} \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}.$$

7. Compute the integral

$$\int_C \frac{x\,dx + y\,dy + z\,dz}{r^3}, \qquad r = \sqrt{x^2 + y^2 + z^2}$$

along the closed path C, which consists of straight-line segments connecting the points $(1,0,0)$, $(1,1,0)$, $(1,1,1)$, $(1,0,1)$, and again $(1,0,0)$.

8. Let

$$w = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

   (a) Find two linearly independent vectors u and v which satisfy $u^T w = 0$ and $v^T w = 0$. (Note: u^T denotes the transpose of u.)

   (b) Show that the set of vectors $\{u, v, w\}$ is linearly independent for the u and v you chose in part (a).

   (c) Consider now the general case where w is a nonzero vector in $\mathbb{R}^3$, and u and v are two linearly independent vectors in $\mathbb{R}^3$ satisfying $u^T w = 0$ and $v^T w = 0$. Show that the three vectors are linearly independent.

9. A lake with volume 10^6 m^3 is found to contain 0.1 kg of mercury. Water is pumped from the lake at 10 m^3/hr, filtered, and returned to the lake at the same flow rate, but with 10^{-9} kg/m^3 of mercury. In addition, clear water from tributaries flows in at 100 m^3/hr, with the well-mixed lake water flowing out at the same rate. Derive and solve a differential equation for the mass of mercury in the lake at any time t.

10. (a) The region R is the portion of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 - 2x = 0$. Compute its volume $V(R)$.

    (b) Sketch the curve C along which the sphere $x^2+y^2+z^2 = 4$ and the cylinder $x^2+y^2-2x = 0$ intersect, and find a suitable parametrization in the form $(x(t), y(t), z(t))$ for it. Also, find the unit tangent vector to the curve C at all the points where such a tangent vector can be found. Are there any points on C where the unit tangent vector to C is not unambiguously defined? If yes, where are these points?

11. The matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

has characteristic equation $\det(A - tI) = -t(t-2)(t+1)$.

(a) Find a matrix S which diagonalizes A. Give the resulting diagonal matrix $D = S^{-1}AS$.

(b) How would you calculate A^7? (Note: You do *not* have to find A^7, just indicate how you would find it efficiently.)

12. (a) If

$$f(x,y) = x^{x^{x^{x^y}}} + (\log x)(\arctan(\arctan(\arctan(\sin(\cos xy))))),$$

find

$$\frac{\partial f}{\partial y}(1,y).$$

Briefly explain your reasoning.

(b) The product of two positive numbers is 1. Choose the numbers so that their sum will be at a minimum.