Preliminary Examination

Department of Mathematical Sciences
Rensselaer Polytechnic Institute
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Instructions:

• The exam contains 12 pages with one problem printed on each page. Please work the problem on the given page. Use the back of the page or include additional pages if necessary. (If you need blank pages for work to be handed in or for scratch just ask.)

• Please submit 10 problems for grading out of the 12 problems given on the exam. You must select at least 3 problems from each group: Calculus (problems 1–4), Linear Algebra (problems 5–8) and Advanced Calculus (problems 9–12).

• Show all work necessary to justify your answers. Answers without proper supporting work will not receive full credit.

• The exam will be graded anonymously. Do not write your name on it. Your names will be attached to your work on the exam only after it has been graded.

• Please write the last 4 digits of your RPI identification number (RIN) at the top right-hand corner of every page you submit for grading. If additional pages are included, then write the problem number at the top left-hand corner and the last 4 digits of your RIN at the top right-hand corner of each additional page.

• Work to be graded should be turned in at the end of the exam. Please place each problem solution face down on one of the twelve numbered envelopes provided. This separation of problems permits fasters and more efficient grading and further enhances anonymity.

• No books, notes, calculators or electronic gizmos of any kind are allowed.

• You have 4 hours to complete the exam.

• Good luck.
1. Let \( \mathcal{P} \) be the plane given by \( x + y + z = -1 \) and let \( \mathcal{C} \) be the curve given parametrically by \( \mathbf{x} = (t^2, t, -t), \ t \in \mathbb{R} \), in the three-dimensional space \( \mathbf{x} = (x, y, z) \).

(a) Show that \( \mathcal{C} \) never intersects \( \mathcal{P} \).

(b) Find the point on \( \mathcal{C} \) that is closest to \( \mathcal{P} \) with respect to Euclidean distance.
2. A tank in the shape of a cylinder is 16 ft high and has an inner radius of 5 ft. The tank is filled with water to a depth of 10 ft. The density of water (by weight) is \( w = 62.5 \text{ lb/ft}^3 \). Formulate and evaluate an integral whose value gives the work (in units of lb \cdot ft) needed to pump all of the water over the top of the tank.
3. Consider the integral

\[ I = \int_0^1 \frac{\cos(x) - 1}{x} \, dx \]

(a) Find a power series of the integrand for evaluating \( I \).

(b) Find an infinite series that gives value of \( I \).

(c) Let \( S \) be the series found in part (b) and let \( S_n \) be the partial sum of \( S \) that includes the leading \( n \) terms in the series. Estimate the error \( |S_n - S| \).
4. Suppose $f(x)$ is a continuous function defined for $x \in \mathbb{R}$. Let $P$ and $Q$ be the points $(a, f(a))$ and $(a + h, f(a + h))$, respectively, for some values of $a$ and $h$.

(a) Derive the equation of the secant line that passes through the points $P$ and $Q$.

(b) Consider the slope of the secant line in part (a). In order for $f(x)$ to be differentiable at $x = a$, what must happen to the slope of the secant line as $h \to 0$? Explain briefly.
5. (a) Let \( V = \{(x, y, z) \in \mathbb{R}^3 : x - 2y + 5z = 0\} \). Show that \( V \) is a subspace of \( \mathbb{R}^3 \) and find a basis for \( V \).

(b) Let
\[
\begin{align*}
\mathbf{u}_1 &= \begin{bmatrix} 1 \\ 2 \\ \alpha \end{bmatrix}, & \mathbf{u}_2 &= \begin{bmatrix} 1 \\ \alpha \\ 2 \end{bmatrix}, & \mathbf{u}_3 &= \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}
\end{align*}
\]

Find all values of \( \alpha \) such that \( \mathbf{u}_1, \mathbf{u}_2 \) and \( \mathbf{u}_3 \) are linearly independent.
6. Consider the vectors $\mathbf{x}^{(n)} \in \mathbb{R}^3$, $n = 0, 1, 2, \ldots$, where the three components of $\mathbf{x}^{(n)}$ are obtained from those of $\mathbf{x}^{(n-1)}$ using the iteration

$$
\begin{align*}
    x_1^{(n)} &= \frac{1}{2} x_2^{(n-1)} \\
    x_2^{(n)} &= \frac{1}{2} x_1^{(n-1)} + \frac{1}{2} x_3^{(n-1)}, \quad n = 1, 2, 3, \ldots, \quad \text{with} \quad x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 1 \\
    x_3^{(n)} &= \frac{1}{2} x_2^{(n-1)}
\end{align*}
$$

(a) Find $\mathbf{x} = \lim_{n \to \infty} \mathbf{x}^{(n)}$ assuming that the iteration converges.

(b) Show that the iteration converges. (Note: it is not enough to compute a few iterates and make the observation that these appear to approach the vector $\mathbf{x}$ found in part (a).)
7. Let

\[
A = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 5 & 2 \\
0 & 2 & 2
\end{bmatrix}
\]

(a) Find an upper triangular matrix \( R \) (with positive elements on the diagonal) such that \( R^T R = A \). (\( R^T \) is the transpose of \( R \).)

(b) Show that \( x^T Ax > 0 \) for all nonzero \( x \in \mathbb{R}^3 \).
8. Let $A$ be an $n \times n$ real symmetric matrix with real eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and corresponding eigenvectors $q_1, q_2, \cdots, q_n$. Assume that $q_i^T q_i = 1$, $i = 1, 2, \ldots, n$.

(a) Show that $q_i^T q_j = 0$ when $i \neq j$.
(b) Find an $n \times n$ orthogonal matrix $Z$ such that $Z^T AZ = D$, where $D$ is a diagonal matrix.
(c) Find a unit vector $u \in \mathbb{R}^n$ such that $u^T Au$ is minimized.
9. Consider the surface $S$ given by $z^2/4 - x^2 - y^2 = 1$.

(a) Is it possible for a tangent plane to $S$ to have a unit normal $n = (2, 2, 1)/3$? Explain.
(b) Is it possible for a tangent plane to $S$ to have a unit normal $n = (1, 1, 1)/\sqrt{3}$? Explain.
10. Find the maximum and minimum of the function \( f(x, y) = 2x^2 + y^2 - y + 3 \) over the region \( x^2 + y^2 \leq 1 \).
11. Evaluate the following integrals:

\[ I_1 = \int_{-1}^{1} \int_{0}^{x^2} xe^{-y^2} \, dy \, dx \quad I_2 = \int_{0}^{3} \int_{0}^{y} \frac{dxdy}{\sqrt{x^2 + y^2}} \]
12. Evaluate the line integral $L = \int_{\Gamma} \mathbf{v} \cdot d\mathbf{x}$ for the following cases:

(a) $\mathbf{v} = (x, x - y, xz)$ with $\Gamma$ the curve $\mathbf{x} = (t^2, 2t, -t)$ from $(0,0,0)$ to $(1,2,-1)$.
(b) $\mathbf{v} = (x^3 - 3y^3, x - 9xy^2)$ with $\Gamma$ the closed curve $x^2 + y^2 = 1$. 