Preliminary Examination

Department of Mathematical Sciences
Rensselaer Polytechnic Institute

February 5, 2011

Instructions:

- Please submit 10 problems for grading out of the 12 problems given on the exam. Do **not** submit solutions for more than 10 problems.
- You must select at least 3 problems from each group: Multi-Variable Calculus (problems 1–4), Linear Algebra (problems 5–8), and Single Variable Calculus (problems 9–12).
- The exam will be graded anonymously. Do **not** write your name on it. Instead, write the last 4 digits of your RPI identification number (RIN) at the top right-hand corner of every page you submit for grading.
- Show all work necessary to justify your answers. Answers without proper supporting work will not receive full credit.
- When turning in your work, place each problem solution face down on its respective numbered envelope at the front of the room.
- No books, notes, calculators or other electronic devices (e.g., iPods) are permitted.
- You have 4 hours to complete the exam.
- Good luck.

ID: ________

1. Let

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4\}$$

and

$$C = \{(x, y, z) \in \mathbb{R}^3 : (x-1)^2 + y^2 \leq 1.\}$$

Use cylindrical coordinates to model and compute the volume V of the intersection of the solid sphere S and the solid cylinder C.

2. Let

$$\mathbf{F}(\mathbf{r}) = \frac{k}{|\mathbf{r}|^2} \frac{\mathbf{r}}{|\mathbf{r}|}, \quad |\mathbf{r}| > 0$$

where $\mathbf{r} = (x, y, z)$ and k is a non-zero real constant.

Show that $\mathbf{F}$ is a conservative vector field.

ID: ________

3. A bent wire in the plane assumes the shape of a curve C modeled by

$$\mathbf{r}(t) = (x(t), y(t)) = \left(e^t \cos(t), e^t \sin(t)\right), \quad 0 \le t \le \frac{\pi}{2}.$$

The mass density of the wire $m(x, y)$, at each point (in suitable units) is equal to the distance between the point and the origin. (Mass density is mass per unit length.)

Model and compute the total mass of the wire.

4. A parametric surface in $\mathbb{R}^3$ is modeled by the vector valued function

$$\mathbf{r}(u, v) = (u, \sqrt{u}\cos(v), \sqrt{u}\sin(v)), 0 \leq u \leq 1,\ 0 \leq v \leq 2\pi.$$

Find a reasonable model for the area of the surface and within your model, compute the area.

5. The Fibonacci sequence is defined as

$$x_n = x_{n-1} + x_{n-2}, \quad x_0 = x_1 = 1, \quad n = 2, 3, \cdots .$$

Give a closed form explicit formula for the n^{th} term in the Fibonacci sequence, $n = 0, 1, \cdots$.
Hint: The elements of the Fibonacci sequence satisfy

$$\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix}$$

where $\mathbf{A}$ is an appropriately defined 2 by 2 matrix.

ID: ________

6. With respect to the standard bases, the matrix of a linear transformation from $\mathbb{R}^5$ into $\mathbb{R}^4$ is given by

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 0 & -1 & 3 \\ 3 & 8 & -4 & -5 & 13 \\ 3 & 1 & -2 & -1 & 2 \\ -9 & 4 & 4 & -1 & 5 \end{bmatrix}.$$

(i) Find a basis for the null space of the transformation.

(ii) Find a basis for the range of the transformation.

7. Consider the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that projects every point $\boldsymbol{x} \in \mathbb{R}^2$ onto the line $x_2 = x_1$ and the linear transformation $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that rotates every vector $\boldsymbol{x}$ clockwise about the origin through 90 degrees. Find a matrix representation, with respect to the standard basis, for the linear transformation $F \circ T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

8. Determine all values of x such that the parallelepiped defined by the three vectors $\boldsymbol{r}$, $\boldsymbol{t}$, $\boldsymbol{s} \in \mathbb{R}^3$ has a volume of 5 where $\boldsymbol{r} = [2, 3, -1]$, $\boldsymbol{t} = [1, x, -2]$, and $\boldsymbol{s} = [3, 1, 4]$.

ID: ________

9. A boy, standing on a straight road, is flying a kite. The kite travels directly above the road at a constant height of 300m and has a speed of 5m/sec. Assuming that the string between the boy and the kite is always straight, at what rate is the boy paying out the string, when its length is 500m?

10. Gabriel's horn is obtained by rotating the curve $1/x$, $1 \le x$, about the x axis. Show that this object has finite volume (find it) but infinite area.

11. Compute the limits of the following expressions:

(a) $\lim_{n\to\infty} n\left(x^{1/n} - 1\right)$ for $x > 0$,

(b) $\lim_{n\to\infty} \dfrac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots\sqrt{n}}{n^{3/2}}$.

12. The variables x and y are related by $x = \int_0^y \frac{dt}{\sqrt{1+4t^2}}$. Show that $\frac{d^2y}{dx^2}$ is proportional to y.