Preliminary Examination

Department of Mathematical Sciences
Rensselaer Polytechnic Institute

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Instructions:

- Please submit 10 problems for grading out of the 12 problems given on the exam. Do **not** submit solutions for more than 10 problems.
- The exam will be graded anonymously. Do **not** write your name on it. Instead, write the last 4 digits of your RPI identification number (RIN) at the top right-hand corner of every page you submit for grading.
- Show all work necessary to justify your answers. Answers without proper supporting work will not receive full credit.
- When turning in your work, place each problem solution face down on its respective numbered envelope at the front of the room.
- No books, notes, calculators or other electronic devices (e.g., iPods) are permitted.
- You have 4 hours to complete the exam.
- Good luck.

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Total	

1. Suppose A is an invertible symmetric $n \times n$ real matrix for which $\{\mathbf{w}^{(i)}\}_{i=1}^{k}$ are known to be real eigenvectors (but not necessarily the complete set) with corresponding distinct eigenvalues $\{\lambda^{(i)}\}_{i=1}^{k}$:

$$\mathsf{A}\mathbf{w}^{(i)} = \lambda^{(i)}\mathbf{w}^{(i)}.$$

Define

$$\mathsf{B} = \mathsf{A} + \sum_{i=1}^{k} c_i \mathbf{w}^{(i)} \mathbf{w}^{(i)T},$$

where $\{c_i\}_{i=1}^{k}$ are some real constants and $\mathbf{v}^T$ denotes the transpose of vector $\mathbf{v}$.

(a) For what values of $\{c_i\}_{i=1}^{k}$ is B invertible?

(b) When B is invertible, express B^{-1} purely in terms of A^{-1}, the constants $\{c_i\}_{i=1}^{k}$, the eigenvectors $\{\mathbf{w}^{(i)}\}_{i=1}^{k}$, and the eigenvalues $\{\lambda^{(i)}\}_{i=1}^{k}$.

2. Suppose D is an $n \times n$ diagonal matrix with entries $D_{ii} = d_i > 0$, and that B is a $n \times k$ real matrix, with $k < n$. Consider the matrix

$$C = DBB'D$$

where B' denotes the transpose of matrix B.

What is the maximum rank C could have (over all possible choices of D and B subject to the conditions described above)? Explain your reasoning.

3. Suppose that the permutation operator

$$\mathsf{P} : \mathbb{R}^N \to \mathbb{R}^N$$

is defined so that

$$\mathsf{P}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_N \end{bmatrix} = \begin{bmatrix} v_N \\ v_1 \\ v_2 \\ \vdots \\ v_{N-1} \end{bmatrix}.$$

for all $\mathbf{v} \in \mathbb{R}^N$. Suppose further you have found N distinct eigenvalues $\{\lambda_j\}_{j=1}^N$ of P and corresponding normalized column eigenvectors of the form:

$$\mathbf{w}^{(j)} = \frac{1}{\sqrt{N}}\begin{bmatrix} 1 \\ \lambda_j^{-1} \\ \lambda_j^{-2} \\ \vdots \\ \lambda_j^{-(N-2)} \\ \lambda_j^{-(N-1)} \end{bmatrix}, j = 1, \ldots, N.$$

One could solve for $\{\lambda_j\}_{j=1}^N$ explicitly but you don't need to do this for this problem.

(a) Find the eigenvalues and column eigenvectors of the transpose of the permutation operator $\mathsf{P}^{(T)}$ in terms of $\{\lambda_j\}_{j=1}^N$ and $\{\mathbf{w}^{(j)}\}_{j=1}^N$.

(b) Now consider the $N \times N$ matrix:

$$\mathsf{C} = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_{N-1} & c_N \\ c_N & c_1 & c_2 & \cdots & c_{N-2} & c_{N-1} \\ c_{N-1} & c_N & c_1 & \cdots & c_{N-3} & c_{N-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_3 & c_4 & c_5 & \cdots & c_1 & c_2 \\ c_2 & c_3 & c_4 & \cdots & c_N & c_1 \end{bmatrix}$$

Show that $\{\mathbf{w}^{(j)}\}_{j=1}^N$ are column eigenvectors for C, and compute their corresponding eigenvalues in terms of $\{\lambda_j\}_{j=1}^N$, $\{\mathbf{w}^{(j)}\}_{j=1}^N$, and $\{c_j\}_{j=1}^N$.

4. We define subsets $\{S_k\}_{k=1}^{K}$ of the integers as follows:

$$\begin{aligned} S_1 &= \{1, 2, \ldots, M\}, \\ S_2 &= \{M+1, M+2, \ldots, 2M\}, \\ &\vdots \\ S_K &= \{(K-1)M+1, (K-1)M+2, \ldots, KM\} \end{aligned}$$

where K and M are positive integers, and we set $N = KM$. Define the following subspaces of $\mathbb{R}^N$:

$$V_0 = \bigcap_{k=1}^{K} \{(x_1, x_2, \ldots, x_N) : x_i = x_j \text{ when } i, j \in S_k\},$$

$$V_k = \{(x_1, x_2, \ldots, x_N) : x_i = 0 \text{ for } i \notin S_k, \sum_{i=1}^{N} x_i = 0\} \text{ for } k = 1, \ldots, K.$$

(a) Compute the dimension of each subspace $\{V_k\}_{k=0}^{K}$.

(b) Show that the subspaces $\{V_k\}_{k=0}^{K}$ are all orthogonal to each other.

(c) Show that the subspaces $\{V_k\}_{k=0}^{K}$ span $\mathbb{R}^N$.

5. The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin. (Identify which is which.)

6. Find the volume enclosed between the two surfaces $z = 8 - x^2 - y^2$ and $z = x^2 + 3y^2$.

7. Find the surface area of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$.

8. Find the flux $\int_C \boldsymbol{F} \cdot \boldsymbol{n}\, ds$ of the vector field $\boldsymbol{F}(x, y) = 2x\mathbf{i} - 3y\mathbf{j}$ outward across the ellipse C whose parametric form is $x = \cos t$, $y = 4 \sin t$, $0 \leq t \leq 2\pi$. (It is understood that ds represents arc length differential and $\boldsymbol{n}$ represents the outward unit normal.)

9. Newton's method can be used to find a sequence $\{x_n\}_{n=0}^{\infty}$ of decimal approximations to the number $\frac{1}{214}$.

(An advantage over the long division algorithm is that no divisions are necessary.)

Let $x_0 = .005$. Calculate x_1 with no rounding.

Show clearly the recursion formula that you use. Show clearly your (hand) decimal calculations.

10. Let

$$f(x) = \tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right), \quad -\infty < x < \infty, \quad x \neq 0.$$

Find an explicit formula for $f(x)$ that is free of trigonometric functions. Justify your claim.

Hint: think derivative

11. Let a and b be positive real numbers. Find an explicit form for a positive real-valued function, $r(\phi)$, $0 \le \phi < 2\pi$, such that the two sets

$$\mathcal{S}_1 = \left\{(x, y) : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1\right\}$$

and

$$\mathcal{S}_2 = \{(x, y) \in \mathbb{R}^2 : (x, y) = (r(\phi)\cos(\phi), r(\phi)\sin(\phi)), \quad \text{for some } \phi \in [0, 2\pi)\}$$

are equal.

Argue that the two sets are equal.

12. The center of a circle in the x, y-plane, of radius 1, has coordinates $(x, y) = (0, c)$ for some $c > 0$. The circle is tangent to the parabola, $y = x^2$, $-\infty < x < \infty$, at exactly two points. Compute all possible values for c.