SPECIFICATIONS

(TIME)  
STEP RESPONSE:
- overshoot
- damping ratio
- rise time
- settling time
- steady-state error

FREQUENCY RESPONSE:
- phase margin
- gain margin
- closed-loop bandwidth
- $M_p$ (peak bode magnitude)
Fig. 4.5. Transient response of a second-order system (Eq. 4.9) for a step input.

decreases, the closed-loop roots approach the imaginary axis and the response becomes increasingly oscillatory.

The Laplace transform of the unit impulse is \( R(s) = 1 \), and therefore the output for an impulse is

\[
C(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2},
\]

(4.10)

which is \( T(s) = C(s)R(s) \), the transfer function of the closed-loop system. The transient response for an impulse function input is then

\[
c(t) = \frac{\omega_n}{\beta} e^{-\zeta \omega_n t} \sin \omega_n \beta t,
\]

(4.11)

which is simply the derivative of the response to a step input. The impulse response of the second-order system is shown in Fig. 4.6 for several values of the damping ratio, \( \zeta \). Clearly, one is able to select several alternative performance measures from the transient response of the system for either a step or impulse input.

Standard performance measures are usually defined in terms of the step response of a system as shown in Fig. 4.7. The swiftness of the response is measured by the rise time \( T_r \) and the peak time. For underdamped systems with an overshoot, the 0–100% rise time is a useful index. If the system is overdamped, then the peak time is not defined and the 10–90% rise time, \( T_{10} \), is normally used.
The similarity with which the actual response matches the step input is measured by the percent overshoot and settling time $T_s$. The percent overshoot, P.O., is defined as

$$\text{P.O.} = \frac{M_p - 1}{1} \times 100\%$$

(4.12)

for a unit step input, where $M_p$ is the peak value of the time response. The settling time, $T_s$, is defined as the time required for the system to settle within a certain
dominant roots of the second-order system as long as the real part of the dominant roots is less than \( \frac{1}{\zeta} \) of the real part of the third root.

Using a computer simulation, when \( \zeta = 0.45 \) one can determine the response of a system to a unit step input. When \( \gamma = 2.25 \) we find that the response is over-damped since the real part of the complex poles is \(-0.45\), while the real pole is equal to \(-0.444\). The settling time is found via the simulation to be 12.8 seconds. If \( \gamma = 0.90 \) or \( 1/\gamma = 1.11 \) is compared to \( \zeta \omega_n = 0.45 \) of the complex poles we find that the overshoot is 12% and the settling time is 6.4 seconds. If the complex roots were entirely dominant we would expect the overshoot to be 20% and the settling time be \( 4/\zeta \omega_n = 4.4 \) seconds.

Also, we must note that the performance measures of Fig. 4.8 are only correct for a transfer function without finite zeros. If the transfer function of a system possesses finite zeros and they are located relatively near the dominant poles, then these zeros will affect the transient response of the system [5].

The transient response of a system with one zero and two poles may be affected by the location of the zero [5]. The percent overshoot for a step input as a function of \( \alpha/\zeta \omega_n \) is given in Fig. 4.10 for the system transfer function

\[
T(s) = \frac{(\alpha \zeta^2/\gamma)(s + \alpha)}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

The correlation of the time-domain response of a system with the \( s \)-plane location of the poles of the closed-loop transfer function is very useful for selecting the specifications of a system. In order to clearly illustrate the utility of the \( s \)-plane, let us consider a simple example.

![Fig. 4.10. Percent overshoot as a function of \( \zeta \) and \( \omega_n \) when a second-order transfer function contains a zero.](image)
show the selectivity of the ITAE index in comparison with the ISE index. The value of the damping ratio $\xi$ selected on the basis of ITAE is 0.7, which, for a second-order system, results in a swift response to a step with a 5% overshoot.

Example 4.4. The signal-flow graph of a space vehicle attitude control system is shown in Fig. 4.20 [7]. It is desired to select the magnitude of the gain $K_i$ in order to minimize the effect of the disturbance $U(s)$. The disturbance in this case is equivalent to an initial attitude error. The closed-loop transfer function for the disturbance is obtained by using the signal-flow gain formula as

$$\frac{C(s)}{U(s)} = \frac{P_1(s) \Delta_1(s)}{\Delta(s)} = \frac{1}{1 + K_i K_1 s^{-1}} \cdot \frac{s(s + K_1 K_2)}{s^3 + K_1 K_3 s + K_1 K_5 K_p}.$$  \hspace{1cm} (4.43)

Typical values for the constants are $K_i = 0.5$ and $K_1 K_3 K_p = 2.5$. Then the natural frequency of the vehicle is $f_n = \sqrt{2.5 / 2\pi} = 0.25$ cycles/sec. For a unit step disturbance, the minimum ISE can be analytically calculated. The attitude $c(t)$ is

Fig. 4.20. A space vehicle attitude control system.
\[
\frac{C(s)}{R(s)} = \frac{G_c(s) G_p(s)}{1 + G_c(s) G_p(s) H(s)}
\]
ANALYTICAL METHODS

• Root Locus - locus of roots of
  \(1 + G(s)H(s) = 0\) in \(s\)-plane
  as the loop gain varies

• Bode Plot - plots of \(|G(s)H(s)|\)
  and \(\arg[G(s)H(s)]\) versus \(\omega\)

• Nyquist Plot - polar plot of
  \(G(s)H(s)\) with \(\omega\) as a
  parameter

• Nichols Plot - plot of
  \(|G(s)H(s)|\) vs. \(\arg[G(s)H(s)]\)
  with \(\omega\) as a parameter
**Example 10-4 in Dorf**

\[ G_c(s) = 2.65 \left( \frac{s+4}{s+10.6} \right) \times 36.4 \]

\[ \text{lead} \]

\[ \text{gain} \]
**Lead Compensator** [for DC Motor]

\[ G_c(s) = \alpha \left( \frac{s + \alpha Z}{s + Z} \right) \quad \alpha > 1 \]

\[ p = \alpha Z \]

**Purpose:**
- Improve damping
- Reduce rise time
Example 10.2 in Dorf

**Magnitude**

- **K_v = 20**
- Uncomp. (K = 40)

**Phase**

- Comp.
- Uncomp.
- \(-180^\circ\)
- 6.2, 8.4

\[ G_{c(s)} = \frac{3 \left( \frac{s + 4.8}{s + 14.4} \right)}{ \text{lead} } \times 40 \]

**Lead**

**Gain**
**LAG COMPENSATOR**

\[ G_c(s) = \left( \frac{s + \alpha p}{s + p} \right) \quad \alpha > 1 \]

**Purpose:**
- Reduce steady-state error
Position Error Constant

\[ C(s) = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)} \cdot R(s) = \frac{G(s)}{1 + G(s)H(s)} \cdot R(s) \]

\[ E(s) = R(s) - C(s) = \left[ \frac{1 + G_c(s)G_p(s)H(s)}{1 + G_c(s)G_p(s)H(s)} \right] \cdot R(s) = \left[ \frac{1 + G(s)H(s) - G(s)}{1 + G(s)H(s)} \right] \cdot R(s) \]

\[ E(s) = \lim_{t \to \infty} e(t) = \lim_{s \to 0} s \cdot E(s) = \lim_{s \to 0} s \frac{R(s)}{1 + G(s)} = \left\{ 0, \text{constant}, \infty \right\} \]

\[ R(s) = \frac{A}{s} \]  [STEP]

\[ R(s) = \frac{A}{s^2} \]  [RAMP]

\[ R(s) = \frac{A}{s^3} \]  [ACCELERATION]

Velocity Error Constant

\[ e_{ss} = \lim_{s \to 0} \frac{A}{s^3} \quad e_{ss} = \lim_{s \to 0} \frac{A}{sG(s)} \]

\[ K_p \triangleq G(0) \]

\[ K_v \triangleq G(s) \bigg|_{s=0} \]

Defined for Type 0 systems

Defined for Type 0 systems

Defined for Type 0 systems

Acceleration Error Constant

<table>
<thead>
<tr>
<th>Number of integrations in ( G(s) )</th>
<th>Input</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>( \frac{A}{1 + K_p} )</td>
</tr>
<tr>
<td>( A )</td>
<td>( \frac{A}{K_v} )</td>
</tr>
<tr>
<td>( A )</td>
<td>( \frac{A}{K_v} )</td>
</tr>
</tbody>
</table>
Obtaining a Discrete-Time Model

From a Continuous-Time Model

Given:

\[ \begin{align*}
  u(kT) & \xrightarrow{D/A} u(t) & \xrightarrow{G_c(s)} c(t) & \xrightarrow{A/D} c(kT)
\end{align*} \]

Find:

\[ \begin{align*}
  u(kT) & \xrightarrow{G_d(z)} c(kT)
\end{align*} \]
METHODS

1. TIME-DOMAIN
   • See Sec. 13.1 of "LINEAR SYSTEMS" for 1ST ORDER EXAMPLE
   • See Sec. 13.7 for 2ND ORDER EX.

2. TRANSFER FUNCTIONS
   \[ G_d(z) = \sum \text{Res} \left\{ \frac{(z-1)G_c(s)}{(z-e^{-sT})s} \right\} \]
   \[ \text{s=0 & poles of } G_c(s) \]

3. STATE VARIABLES
   \[ \dot{x} = A_c x(t) + B_c \ddot{u}(t) \]
   \[ x(k+1) = A_d x(k) + B_d u(k) \]
   where:
   \[ A_d = e^{A_c T} \]
   \[ B_d = \int_0^T e^{A_c \tau} d\tau \cdot B_c \]

4. TUSTIN APPROX.: DIGITAL APPROX. OF ANALOG CONTROL
Modelling Example - Transfer Function

\[ \ddot{c} + \dot{c} = \ddot{u}(t) \quad \Rightarrow \quad G_c(s) = \frac{1}{s(s+1)} \]

\[ G_d(z) = \sum \text{Res} \left\{ \frac{z-1}{(z-e^{-sT}) s^2 (s+1)} \right\}_{s=0, -1} \]

\[ = (z-1) \left\{ \frac{d}{ds} \left[ \frac{1}{(z-e^{-sT})(s+1)} \right]_{s=0} + \left[ \frac{1}{(z-e^{-sT}) s^2} \right]_{s=-1} \right\} \]

\[ G_d(z) = \frac{(e^{-T}-1+T) z + 1 - (1+T)e^{-T}}{(z-1)(z-e^{-T})} \]

For \( T=1 \),

\[ G_d(z) = \frac{0.368 \, z + 0.264}{(z-1)(z-0.368)} \]

\[ \xrightarrow{\text{Find } D(z)} \]

STEP INPUT SETTLE IN

T
2T
3T

1. Minimal Prototype
2. Ripple Free (Finite Settling Time)
3. Ripple Free with Control-Magnitude Constraints
Modelling Example - State Variables

\[
\frac{d}{dt} \begin{bmatrix} \dot{c}_c \\ \dot{c}_c \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{c}_c \\ \dot{c}_c \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tilde{u}(t)
\]

\(A_c\)
\(B_c\)

gives

\[
A_d = \begin{bmatrix} e^{-T} & 0 \\ 1-e^{-T} & 1 \end{bmatrix}
\]

\[
B_d = \begin{bmatrix} 1-e^{-T} \\ T-1+e^{-T} \end{bmatrix}
\]

for \(T = 1\),

\[
A_d = \begin{bmatrix} 0.368 & 0 \\ 0.632 & 1 \end{bmatrix} \quad \text{and} \quad B_d = \begin{bmatrix} 0.632 \\ 0.368 \end{bmatrix}
\]

(see C&M, page 21)
**DESIGN METHODS**

1. **Minimal Prototype**
   - Zero error at sampling instants after minimal number of sample times
   
   \[
   \begin{align*}
   c(t) &= R \quad \text{for } t \geq t_1 = 70 \\
   \dot{c}(t) &= 0 \quad t = T
   \end{align*}
   \]
   CAN'T SATISFY BOTH :: $\dot{c}(t) \neq 0$

2. **Ripple-Free Response (Finite Settling Time)**
   - Zero error for all time after some number of sample times
   
   \[
   \begin{align*}
   c(t) &= R \quad \text{for } t \geq t_1 \geq 0 \\
   \dot{c}(t) &= 0 \quad t_1 = 2T
   \end{align*}
   \]

   NOT VERY USEFUL BUT EASY TO IMPLEMENT

   MAY REQUIRE LARGE CONTROL GAINS (OUTPUTS) ESPECIALLY IF $T \to 0$
**Minimal Prototype - Frequency Response**

\[
\frac{C(z)}{R(z)} = \frac{D(z) G(z)}{1 + D(z) G(z)}
\]

For 2nd order system & step input,

\[
\frac{C}{R} = z^{-1}
\]

(Generic response: delay of T in step)

Solving for \( D(z) \):

\[
\frac{DG}{1 + DG} = z^{-1} \quad \Rightarrow \quad D(z) = \frac{z^{-1}}{(1-z^{-1})G(z)}
\]

or

\[
D(z) = \frac{1}{(z-1)G(z)}
\]
Minimal Prototype - Time Response

Example (see C & M, page 249)

\[ G(s) = \frac{1}{s(s+1)} \] and \[ T = 1 \]

In general,

\[
\begin{bmatrix}
\dot{c}(k+1) \\
C(k+1)
\end{bmatrix} = \begin{bmatrix}
0.368 & 0 \\
0.632 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{c}(k) \\
C(k)
\end{bmatrix} + \begin{bmatrix}
0.632 \\
0.368
\end{bmatrix} u(k)
\]

Using ICs \( C(0) = \dot{C}(0) = 0 \) and \( C(1) = 1 \) \( \leftarrow \) Required for Min. Proto.

For \( k = 0 \) i.e.

\[
\begin{bmatrix}
\dot{c}(1) \\
1
\end{bmatrix} = A \begin{bmatrix}
0 \\
0
\end{bmatrix} + B u(0)
\]

\[ u(0) = \frac{1}{0.368} = 2.72 \quad (\text{or} \ 2.718) \]

\[ \dot{c}(1) = 0.632 \quad u(0) = 1.72 \quad (\text{or} \ 1.718) \]
for \( k = 1 \): 
\[
\begin{bmatrix}
\dot{c}(2) \\
1
\end{bmatrix} = A \begin{bmatrix}
1.72 \\
1
\end{bmatrix} + B u(1)
\]

\[\Rightarrow u(1) = -2.95 \quad \text{and} \quad \dot{c}(2) = -1.225\]

Will get 
\[u(k) = 2.72, -2.95, 2.12, \ldots\]

so 
\[u(z) = 2.72 - 2.95 z^{-1} + 2.12 z^{-2} + \ldots\]

Since 
\[e(k) = 1, 0, 0, \ldots\]

\[E(z) = 1\]

Thus, 
\[D(z) = \frac{U(z)}{E(z)}\]

\[= 2.72 - 2.95 z^{-1} + 2.12 z^{-2} + \ldots\]

which can be shown to be equivalent to 
\[
\frac{2.718 z - 1}{z + 0.718}
\]
MINIMAL PROTOTYPE

EXAMPLE (see C&M, page 272)

\[ G(s) = \frac{1}{s(s+1)} \quad \text{and} \quad T = 1 \]

\[ \Rightarrow \quad G(z) = \frac{0.368 z + 0.264}{(z-1)(z-0.368)} \]

Thus,

\[ D(z) = \frac{z - 0.368}{0.368 z + 0.264} = \frac{2.718 z - 1}{z + 0.718} \]

Since \( D(z) = \frac{U(z)}{E(z)} \), the control law is

\[ u(k+1) + 0.718 u(k) = 2.718 e(k+1) - e(k) \]

or

\[ u(k) = -0.718 u(k-1) + 2.718 e(k) - e(k-1) \]

IN GENERAL:

\[ D(z) = \frac{K(z)}{z \{ G_0(z) G(z) \} [1 - K(z)]} \]

WHERE

<table>
<thead>
<tr>
<th>INPUT</th>
<th>K(z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>STEP</td>
<td>( z^{-1} )</td>
</tr>
<tr>
<td>RAMP</td>
<td>( 2 z^{-1} - z^{-2} )</td>
</tr>
<tr>
<td>ACCEL</td>
<td>( 3z^{-2} - 3z^{-2} + z^{-3} )</td>
</tr>
</tbody>
</table>
RIPPLE-FREE DESIGN

Example (see C&M, page 252)

Allow 2 steps for $c(k) = 1$ but require that $\dot{c}(k) = 0$

for $k=0$:

$$\begin{bmatrix} \dot{c}(1) \\ c(1) \end{bmatrix} = A \begin{bmatrix} 0 \\ 0 \end{bmatrix} + B \ u(0)$$

for $k=1$:

$$\begin{bmatrix} \dot{c}(2) \\ c(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A \begin{bmatrix} \dot{c}(1) \\ c(1) \end{bmatrix} + B \ u(1)$$

⇒ 4 equations in the 4 unknowns: $c(1), \dot{c}(1), u(0), u(1)$

Solve for $u(0) = 1.582$

$u(1) = -0.58$

and $u(k) = 0$ for $k \geq 2$
Also \[ e(0) = 1 \]
\[ E(1) = 1 - (1) = 1 - 0.582 = 0.418 \]

Thus \[ U(z) = 1.582 - 0.58 z^{-1} \]
\[ E(z) = 1 + 0.418 z^{-1} \]

So \[ D(z) = \frac{1.582 - 0.58 z^{-1}}{1 + 0.418 z^{-1}} = \frac{U(z)}{E(z)} \]
\[ = \frac{1.582 z - 0.58}{z + 0.418} \]

1.582 \( e(k+1) \) - 0.58 \( e(k) \) = \( u(k+1) \) + 0.418 \( u(k) \)

FST \[ \rightarrow \]
\[ u(k) = 1.582 e(k) - 0.58 e(k-1) - 0.418 u(k-1) \]

**Note:** Many controllers for various performance criteria result in \( D(z) \) with the form \[ A \frac{z + \rho}{z + \rho} \]
Tustin Approximation

\[ y(t) = \int_0^t x(t') \, dt' \]

\[ y(kT + T) = \int_0^{(k+1)T} x(t') \, dt' \]

\[ = \int_0^{kT} x(t') \, dt' + \int_{kT}^{(k+1)T} x(t') \, dt' \]

\[ = y(kT) + \int_{kT}^{(k+1)T} x(t) \, dt \]

\[ y(kT + T) = y(kT) + \frac{T}{2} \left[ x(kT + T) + x(kT) \right] \]

\[ \bar{Y}(z) \bar{Z} = \bar{Y}(z) + \frac{T}{2} \left[ \bar{X}(z) \bar{Z} + \bar{X}(z) \right] \]

\[ \frac{\bar{Y}(z)}{\bar{X}(z)} = \frac{T}{2} \frac{z+1}{z-1} \approx \frac{1}{5} \quad \text{(Integration)} \]

\[ \therefore s = \frac{2}{T} \frac{z-1}{z+1} \quad \text{• Form of Bilinear Transformation (Scaled)} \]