Research Summary

Fengyan Li

Computational science nowadays forms the third pillar in science, next to theory and experiment. My general research interests are in the design, analysis and implementation of numerical methods for solving differential equations arising in sciences and engineering. The research I have been working on mainly contains three different but related subjects: the local-structure-preserving high-resolution discontinuous Galerkin methods, the computational electromagnetism, and the efficient and accurate computational methods for mathematical models in computer vision and optimal control. In the following, I will describe in detail these subjects as well as the research work I have been working related to each topic.

1 Local-Structure-Preserving High-Resolution Discontinuous Galerkin Methods

With the rapid development of the modern science and technology, there is an increasing need for the highly accurate and highly stable methods in many applications in science and engineering. The discontinuous Galerkin method (DGM) is one family of such methods. The main feature of the method is the use of the completely discontinuous piecewise smooth functions to approximate solutions. These robust, compact, locally conservative, and highly accurate methods can easily handle complex geometric domains, variety of boundary conditions and irregular meshes, therefore they are ideal for the hp-adaptive strategies and the parallel implementation. These advantages make the methods rapidly gain their popularity [1] in many applications such as weather-forecasting, image processing, aircraft design, pollution control, medical equipment design, oil recovery simulation, among many others.

It is known that by allowing discontinuities in the approximating functions, DGMs involves more unknowns compared with the traditional methods using continuous functions as approximations. On the other hand, these "extra" unknowns provide algorithm developers opportunities to design stable and accurate schemes by properly choosing the inter-element treatment (also called numerical fluxes). This issue has been pursued by many researchers regarding different problems. There is also another issue, which just starts drawing researchers' attention, that is, the discontinuities of the solution spaces also provide flexibility for one to choose the approximation functions within each element. And this is far from being trivial for traditional methods using continuous approximating functions. *This is how my work comes into the development of the discontinuous Galerkin methods.* With my collaborators, I have been working on the

[1] *This increasing popularity can also been seen from the growing size of the participants in the mini-symposia in DGMs at the major national and international scientific events which draw computational engineers and scientists worldwide from government, academia, and industry in recent years. For instance, in the 7th World Congress on Computational Mechanics held in Los Angeles, California, July 16-22, 2006, there were 10 sessions in the DGM mini-symposium and more than 50 top researchers presented their most recent work.*

design of the local-structure-preserving DGMs by extensively exploring this flexibility for wide range of applications. The methods share the similar framework as the traditional DGMs, and the *distinctive feature* of these local-structure-preserving DGMs is the adoption of the approximate solutions that preserve certain structures of the exact solutions within each element, for example the divergence-free property of the magnetic field in Maxwell equations. As a byproduct, the number of the unknowns involved in the final system is reduced so is the computational complexity. The approximation properties of the chosen spaces guarantee no loss of the accuracy.

So far the local-structure-preserving idea in discontinuous Galerkin methods has been investigated in the following applications:

Application 1: time-domain Maxwell equations. By working with Professor Chi-Wang Shu at Brown University and Professor Bernardo Cockburn at University of Minnesota, the local-structure-preserving idea was first explored in solving the time-domain Maxwell equations [5]. Since the magnetic field in the Maxwell equations has the property of being divergence-free when the divergence-free source term is applied, in this project, we use the locally divergence-free functions to approximate the magnetic field within each element. Both the analytical and numerical results demonstrate that the methods are highly accurate and robust, and the computational complexity is greatly reduced.

Application 2: Magnetohydradynamics (MHD) Equations. The second application of the local-structure-preserving idea is in solving the ideal MHD equations [8]. MHD equations study the dynamics of electrically conducting fluids, such as plasmas and liquid metals. *The difficulty* in numerically solving this system comes from the nonlinearity of the problem which explains the complicated behavior of the solutions and also from the instability caused by the numerical nonzero divergence of the magnetic fields. Numerically simulating the behavior of the solutions is crucial for establishing scientific foundation for an integrated fusion simulation in the future. Similar as in the Maxwell equations, our method preserves the *divergence-free property* of the magnetic field within each element. Extensive numerical experiments demonstrate that the use of the locally divergence-free approximations will enhance the numerical stability and reduce some nonphysical features in the solutions without the loss of the accuracy.

Application 3: Hamilton-Jacobi Equations. Hamilton-Jacobi equations often appear in computer vision, image processing, robotic path planning and geographical data processing. When the local-structure-preserving idea is applied to these equations [9], it leads to a reinterpretation and simplified implementation of the DGMs for Hamilton-Jacobi equations developed in [6]. By this reinterpretation, numerical solutions automatically satisfy the *curl-free property* of the exact solutions inside each element. This new reinterpretation allows a more natural framework for stability analysis, and it renders a significantly simplified implementation with reduced computational cost.

Application 4: Laplace Equation. Another application of the local-structure-preserving idea is to solve the model elliptic problem – the Laplace equation in

[7]. In this case, the *piecewise harmonic polynomials*, which satisfy the partial differential equation exactly inside each element, are used to approximate solutions. An exciting observation is, the dimension of this local-structure-preserving solution space depends on the polynomial degree k *linearly*, whereas the dimension of the traditional choice of the solution space (piecewise polynomials) depends on k *quadratically*. This leads to a dramatic reduction in the computational complexity, yet the numerical approximations with the similar resolution are obtained as those by the standard DGMs.

Application 5: frequency-domain Maxwell equations; Application 6: Maxwell eigenproblems. Similar as in the time-domain Maxwell equations, the divergence-free constraints in the magnetic and/or the electric fields are imposed locally inside each element when solving both the frequency-domain Maxwell equations and the Maxwell eigenproblems. The details for both application 5 and 6 as well as the background information will be described in next section.

To summarize, the local-structure-preserving discontinuous Galerkin methods fully use the flexibility in choosing the approximation functions in each local patch of the computational domain. In all the applications mentioned above, the polynomial functions are used due to its easy operation. In some other applications, functions other than polynomials may be better choices, for instance singular functions of certain forms will better resolve the corner singularities around the crack, and this will be studied in forthcoming projects. My accomplishment in this subject gains the recognition of other researchers and I successfully received the *NSF grant award DMS-0652481: On Local-Structure-Preserving Discontinuous Galerkin Methods, June 2006-May 2009.*

2 Computational Electromagnetism

In 1873 the Scottish mathematician and theoretical physicist James C. Maxwell founded the modern theory of electromagnetism with the publication of his "Treatise on Electricity and Magnetism", in which he formulated the equations named Maxwell's equations. And these equations express the basic laws of electricity and magnetism in a unified fashion. Since them, Maxwell equations have been an active research subject in many fields which include the computational mathematics.

The challenges that arise in computational electromagnetism could originate with the equations themselves, such as the divergence-free constraints on the magnetic and/or the electric fields, the possible high-frequency nature in the wave propagation, the challenges could also originate with the data of the problem, i.e. geometric complexity, large size in terms of the characteristic wavelength, inhomogeneous discontinuous materials, and the existence of singularities related to the geometric boundaries or material interfaces. Applications with such characteristics can be found throughout the applied sciences and engineering, such as in the design and analysis of radars and antennas, high-speed electronics and integrated optics, medical equipment design, and wireless communication. Therefore robust and flexible methods are preferred in order to simulate the electromagnetic problems. One choice is the finite element methods in the broad sense. Depending on the objectives, the finite element methods could be the classical finite element methods (approximations are *continuous* across element interfaces), the nonconforming finite element methods (approximations are *continuous at certain points* across element interfaces), and the discontinuous Galerkin methods (approximations are completely *discontinuous* across element interfaces).

So far I have been working with different finite element methods for the following four problems in computational electromagnetism:

- Time-domain Maxwell equations. The original Maxwell equations are a system of time-dependent equations. In the cases when the electromagnetic waves are considered over long time, at relatively high frequency, in complicated domains or in the discontinuous materials, the discontinuous Galerkin method will be a good choice for numerically simulating the behavior of the solutions. In [5], we formulate a discontinuous Galerkin method for time-domain Maxwell equations, the *novel* component in this work is the use of the locally divergence-free functions to approximate the divergence-free magnetic field, and the methods demonstrate the highly accurate and stable performance in numerical experiments. This work is described in detail as Application 1 in Section 1.

- Frequency-domain Maxwell equations. When the electromagnetic wave at certain frequency is concerned, the Maxwell equations can be simplified as the frequency-domain Maxwell equations, and the leading term in the second order version of the equations is a curl-curl operator. Partially motivated by the fact that the curl-curl operator behaves differently when it is applied to the divergence-free component and the gradient component in the Helmholtz decomposition of the solution of Maxwell equations, we decompose the frequency-domain Maxwell equations into two systems [1], among which the relative difficult one is the one solving the divergence-free component of the solution of the original problem. For this new system, I propose three numerical methods [1, 2, 3] by working with Professor Susanne Brenner and Professor Li-yeng Sung at Louisiana State University. Two of them [1, 3] use the nonconforming finite elements approximations, the other [2] uses discontinuous Galerkin type approximations. The divergence-free constraints in the solutions are imposed either by using the locally divergence-free approximations [1, 2], or by using a weighted divergence-term in the bilinear form [3]. The optimal error estimates are established which are verified by the numerical examples, and the analysis by itself provides some new frameworks for studying such problems. The corner singularities are also optimally resolved around the geometric corners by using the graded meshes.

- Maxwell eigenproblems. The spectrum of an operator in general provides the least but significant information about the problem itself, therefore computing the eigenspectrum of the Maxwell problems is of fundamental importance in computational electromagnetism. *One numerical difficulty in this problem is the divergence-free constraint in the eigenfunctions.* The common practice is to neglect this constraint, and the immediate consequence of this is to introduce non-physical eigenvalues into the physical spectrum, and this also causes the loss of the compactness nature of the involved operator which often complicates the analysis. The schemes for the source problems in [1, 2, 3] naturally define three solvers for Maxwell eigenproblems [4]. The *novel* feature of these solvers is to work with the divergence-free functions directly. Not like many other Maxwell eigensolvers based on the full frequency-domain Maxwell equations, the compactness of the involved operator and the uniform error estimates for the source problems greatly simplify the analysis of our proposed eigensolvers. Moreover, these solvers are free of spurious eigenmodes and are free of penalty parameters. We also seek the numerical evidence to detect the spurious eigenmodes in the finite element method setting, explore such evidence through numerical experiments using the variants of the proposed eigensolvers as examples. And this is a very important issue in practice.

- Maxwell equations in exterior domains. Scattering problem is another interesting subject in computational electromagnetism, and *one main challenge comes from the unboundedness of the computational domain*. When the characteristic velocity is much smaller than the light velocity, the Maxwell equations can be approximated by the Darwin model up to at least second order accuracy. In [11], the scattering problem in electromagnetism based on the Darwin model is considered. And by using certain symmetry of the problem, an infinite element method is formulated and analyzed *which uses infinite many elements with finite size*. The final system is in the similar size as those from the discretization of the problem on bounded domains. Numerical experiments agree with the analytical results, and no artificial boundary condition is needed.

3 Computational Methods for Mathematical Models in Computer Vision and Optimal Control

Since the first computer was built up in 1945, computers has brought revolutionary changes to many technologies, one of them is the movie-making. Important mathematical models in the area of computer vision, computer graphics, computer aided design (CAD) and optimal control are a class of partial differential equations: Hamilton-Jacobi equations. Efficiently and accurately solving these equations is crucial in practice. *The main challenge* to solve the Hamilton-Jacobi equations is the non-smoothness in the solution which is related to the nonlinearity of the problem hence the intersection of the characteristic curves. Methods without special caution might produce solutions with oscillations or even produce non-physical solutions. My primary work in this area contains the following two parts:

- High-resolution discontinuous Galerkin methods for time-dependent Hamilton-Jacobi equations. Hamilton-Jacobi equations are closely related to the hyperbolic conservation laws, hence the discontinuous Galerkin methods which have made great success in solving the hyperbolic conversation laws naturally find their ways in solving Hamilton-Jacobi equations. The methods were formulated in [6] by Hu and Shu with a straightforward application of the discontinuous Galerkin methods to the equations satisfied by the gradient of the solution to the Hamilton-Jacobi equations, a least squares process is used to close the system and the constant in the solution is recovered by an integration procedure. In [9], we further reinterpret and simplify several components of the methods *by using the intrinsic structure of the solution*, and this provides a more thorough understanding to the discontinuous Galerkin methods for this problem. And more properties of the our method are summarized as Application 3 in Section 1.

- High-resolution fast sweeping methods for static Hamilton-Jacobi equations. Quite often we encounter systems of nonlinear equations which are derived from the discretization of certain nonlinear differential equations, and they require efficient solvers. The fast sweeping method is one such method with static Hamilton-Jacobi equations as one of its applications. The method follows the causality along the characteristics in a parallel way by combining with the Gauss-Seidal iterations with alternating sweeping directions, and it has the optimal computational complexity in the sense that the number of the iterations is independent of the number of the unknowns. This makes the method extremely useful in real-time application such as the medical imaging.

  The original fast sweeping method for Hamilton-Jacobi equations is finite difference based and it is only first order accurate, yet in many applications, better resolution in the solution is needed. In [10], we formulate a second order fast sweeping method based on discontinuous Galerkin discretization for solving an important family of static Hamilton-Jacobi equations: the Eikonal equations. Numerical experiments demonstrate the second order accuracy and the optimal computational complexity. So far our work defines *the first successful algorithm* in combining the fast sweeping idea and the finite element discretization (discontinuous Galerkin discretization in particular) to solve the static Hamilton-Jacobi type equations.

References

[1] S. C. Brenner, F. Li and L.-Y. Sung, *A locally divergence-free nonconforming finite element method for the reduced time-harmonic Maxwell equations*, Mathematics of Computation, v76 (2007), pp.573-595.

[2] S. C. Brenner, F. Li and L.-Y. Sung, *A locally divergence-free interior penalty method for two-dimensional curl-curl problems*, submitted to SIAM Journal on Numerical Analysis, to appear.

[3] S. C. Brenner, F. Li and L.-Y. Sung, *A nonconforming penalty method for two dimensional curl-curl problems*, in preparation.

[4] S. C. Brenner, F. Li and L.-Y. Sung, *Parameter free nonconforming Maxwell eigensolvers without spurious eigenmodes*, in preparation.

[5] B. Cockburn, F. Li and C.-W. Shu, *Locally divergence-free discontinuous Galerkin methods for the Maxwell equations*, Journal of Computational Physics, v194 (2004), pp.588-610.

[6] C. Hu and C.-W. Shu, *A discontinuous Galerkin finite element method for Hamilton-Jacobi equations*, SIAM Journal on Scientific Computing, v21 (1999), pp.666-690.

[7] F. Li and C.-W. Shu, *A local-structure-preserving local discontinuous Galerkin method for the Laplace equation*, submitted to Methods and Applications of Analysis.

[8] F. Li and C.-W. Shu, *Locally divergence-free discontinuous Galerkin methods for MHD equations*, Journal of Scientific Computing, v22-23 (2005), pp.413-442.

[9] F. Li and C.-W. Shu, *Reinterpretation and simplified implementation of a discontinuous Galerkin method for Hamilton-Jacobi equations*, Applied Mathematics Letters, v18 (2005), pp.1204-1209.

[10] F. Li, C.-W. Shu, Y.-T. Zhang and H.-K. Zhao, *A second order DGM based fast sweeping method for Eikonal equations*, submitted.

[11] L.-A. Ying and F. Li, *Exterior Problem of the Darwin Model and its Numerical Computation*, Mathematical Modelling and Numerical Analysis, v37 (2003), pp.515-532.