Functions of Random Variables

Lecture 19

November 18

Functions of Random Variable

You know distribution of X and Y. Want to know distribution of function of X and Y like $X+Y$, X^2, e^{X+Y}, $g(X,Y)$

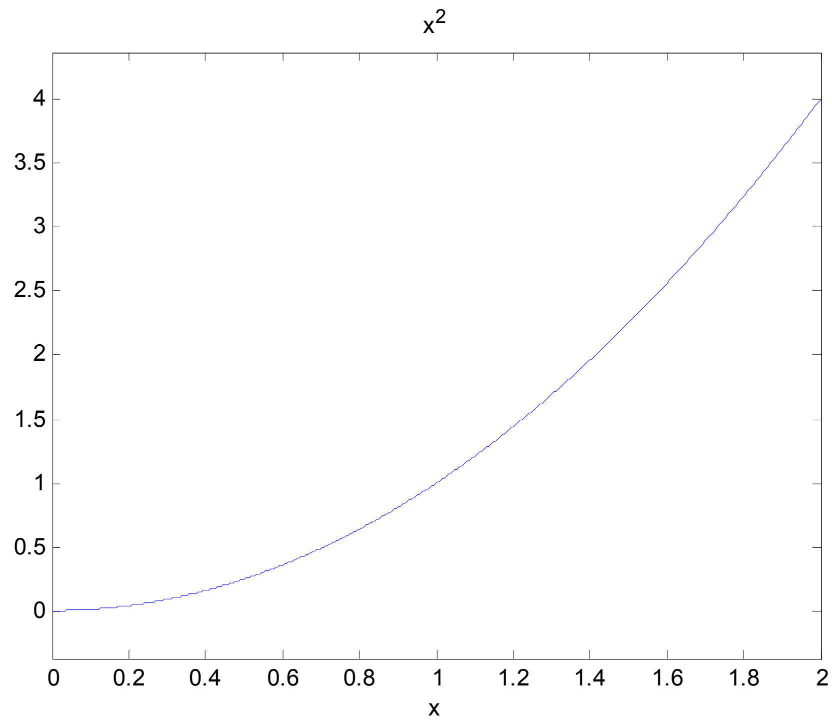
Many ways to attach the problem

1) Discrete case: calculate directly.
2) Method of m.g.f
3) Method of c.d.f
4) Method of transforms

Continuous case

X~uniform(0,2)

$U=X^2$



What is value set of U?

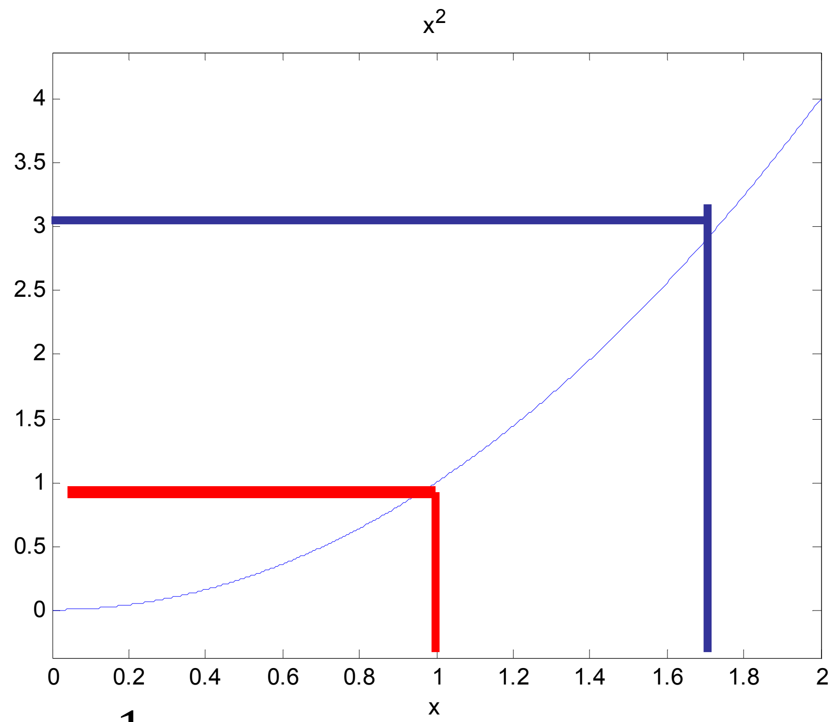
Continuous case

X~uniform(0,2)

U=X²

What is P(U≤1)?

P(U ≤3)?



$$P(U \le 1) = P(X \le 1) = F_X(1) = \int_{x=0}^{1} \frac{1}{2}\,dx = \frac{1}{2}$$

$$P(U \le 3) = P(X \le \sqrt{3}) = F_X(\sqrt{3}) = \int_{x=0}^{\sqrt{3}} \frac{1}{2}\,dx = \frac{\sqrt{3}}{2} = 0.886$$

Continuous case

What is cdf of U?

Calculate F(c)=P(U≤c)

$$F_U(c) = P(U \le c) = P(X \le \sqrt{c}) = F_X(\sqrt{c}) = \int_0^{\sqrt{c}} \frac{1}{2}dx = \begin{cases} 0 & c <= 0 \\ \dfrac{\sqrt{c}}{2} & 0 < c < 4 \\ 1 & c \ge 4 \end{cases}$$

What is pdf of U?

$$f_U(u) = F_U\,'(u) = \frac{d(u^{1/2}/2)}{du} = \frac{u^{-1/2}}{4}$$

$$f_U(u) = \begin{cases} \dfrac{1}{4\sqrt{u}} & 0 < u < 4 \\ 0 & o.w. \end{cases}$$

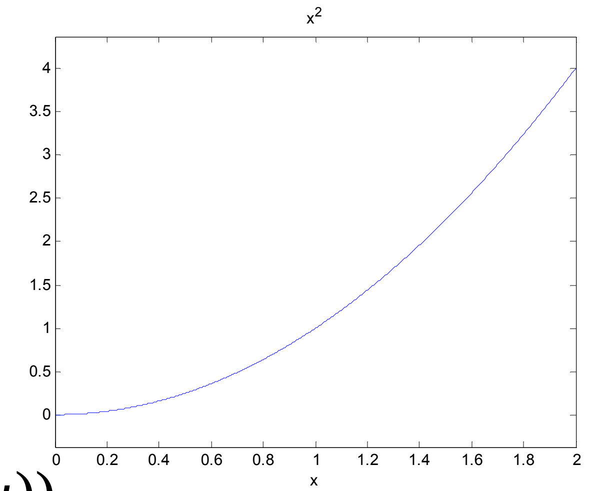
Method of cdf

Basic idea: If $U=g(x)$, Find cdf of U as a function of cdf of X. Differentiate.

Simplifications possible:

Consider case of 1 to 1 functions.

1 to 1 increasing functions



If U is 1 to 1 increasing function

u=g(x) →x=g^{-1}(u)=h(u)

Then the cdf of U is given by

$$F_U(c) = P(g(X) \le u) = P(X \le g^{-1}(u)) = F_X(h(u))$$

And the pdf of U is given by

$$f_U(u) = \frac{dF_X(h(u))}{du} = f_X(h(u))\frac{d(h(u))}{du}$$

by chain rule

No need to explicitly make cdf!

1 to 1 increasing

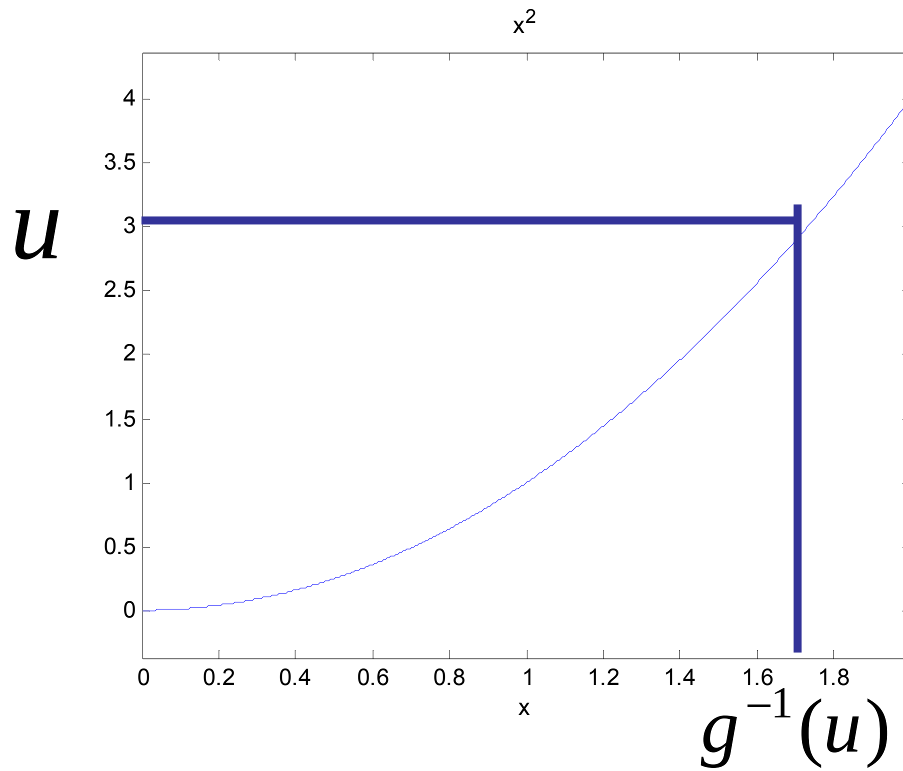
X~uniform(0,2)

$U=X^2$

Inverse

$$g^{-1}(u)=h(u)=\sqrt{u}$$

$$\frac{d(h(u))}{du}=\frac{u^{-1/2}}{2}$$



pdf

$$f_U(u)=f_X(h(u))\frac{d(h(u))}{du}=\frac{1}{2}\frac{u^{-1/2}}{2}=\frac{u^{-1/2}}{4} \qquad 0<u<4$$

You try

Let Z be a standard normal with mean 0 and std dev. 1.

Find pdf of U=2Z+3, using the method just shown. Is this a 1-1 increasing function?

$$h(u) =$$

$$\frac{d(h(u))}{du} =$$

$$f_u(u) = f_Z(h(u))\frac{d(h(u))}{du} =$$

Decreasing function

Let Z be a standard normal with mean 0 and std dev. 1.

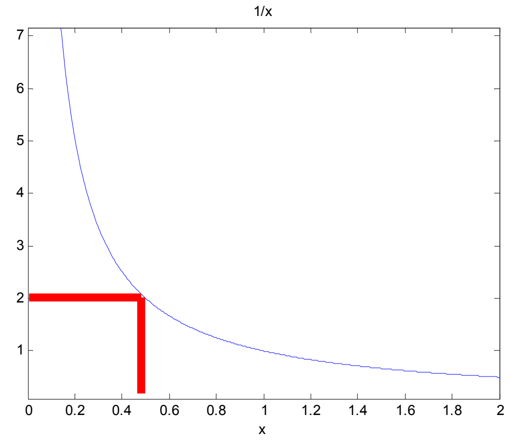
Find pdf of $U=-2Z+3$, using the method just shown.   This is a 1-1 decreasing function!

1 to 1 decreasing functions

If U is 1 to 1 decreasing func

u=g(x) →x=g^{-1}(u)=h(u)

Then the cdf of U is given by



$$F_U(c) = P(g(X) \le u) = P(X \ge g^{-1}(u)) = 1 - F_X(h(u))$$

And the pdf of U is given by

$$f_U(u) = \frac{d(1 - F_X(h(u)))}{du} = f(h(u))\left[-\frac{d(h(u))}{du}\right]$$

by chain rule

In general

Increasing functions

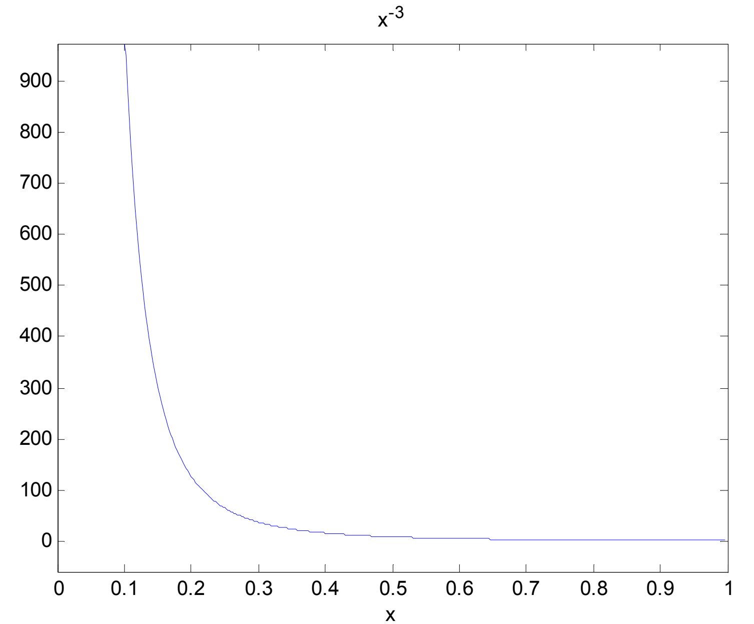
$$\frac{d(h(u))}{du} \geq 0$$

Decreasing functions $\frac{d(h(u))}{du} \leq 0$

So for general 1 to 1 functions, the pdf is

$$f_U(u) = \frac{d(F_U(u))}{du} = f(h(u))\left|\frac{d(h(u))}{du}\right|$$

Example



Let $f_X(x) = 3x^2 \qquad 0 < x < 1$

Find pdf of $U = X^{-3}$

Value set U in $(1,\infty)$

$$h(u) = u^{-1/3}$$

$$\frac{d(h(u))}{du} = -\frac{1}{3}u^{-4/3}$$

$$f_u(u) = f(h(u))\left|\frac{d(h(u))}{du}\right| = 3(u^{-1/3})^2\frac{1}{3}u^{-4/3} = u^{-2} \qquad 1 < u$$

Joint Transforms

If X is a vector of continuous R.V. $(x_1, x_2, \ldots, x_n)$

If U=G(X) is 1 to 1 and the inverse function $x=g^{-1}(u)=h(u)$ is differentiable, then the pdf of U is given by

$$f_U(u) = f_X(h(u))\left|J(u)\right|$$

Where |J(u)| is the determinant of the Jacobian

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} \end{vmatrix}$$

Example

Let

$$f_{XY}(x, y) = e^{-(X+Y)} \qquad 0 < X < \infty \quad 0 < Y < \infty$$

Find pdf

$$U = X + Y$$

- Step 0 – figure out value set $0<U$
- Step 1: find 1 to 1 transform

  $U=X+Y$

  $V=Y$

In general you can use any function (simple as possible) such that $u=a(x,y)$, $v=b(X,Y)$ is 1 to 1

- Step 2 – Solve for (x,y) in terms of (u,v)

$$U=X+Y$$

$$V=Y$$

$$Y=V \qquad X=U-V$$

- Step 3 Calculate the range of U and V

$$0<X<\infty \quad \rightarrow \quad 0<U-V<\infty$$

$$0<Y<\infty \rightarrow \quad 0<V<\infty$$

$$0<V<U<\infty$$

- Step 4 – Calculate Jacobian of

Y=V X=U-V

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = |(1)(1)-(0)(-1)| = |1|$$

Step 5 and 6

- Use formula to calculate the joint of U and V

$$f_{U,V}(u,v) = f_X(u-v,v)\,|\,J\,| = e^{-(u-v+v)}(1) \quad 0 < V < U < \infty$$

- Calculate marginal of U

$$f_U(u) = \int_0^U e^{-(u)}dv = ue^{-u} \qquad 0 < u < \infty$$