Lecture 4: Conditional Probability

Probability Theory and Applications

Fall 2005

September 9

Conditional Probability

The conditional probability of A given B has occurred

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(AB)}{P(B)}$$

Motivation

- Consider male population classified according to lung disorder present (A) and smoker (B). Select person at random.
- What is prob that person smokes? P(B)=25/100
- What is prob that person has lung disorder?

  P(A)=11/100

	Smoker B	Non-smoker $\overline{B}$	
Dis-order present A	5	6	11
Dis-order absent	20	69	89
	25	75	

Given person smokes, what is prob of disorder? P(A|B)= P(AB)/P(B)=5/25=.2

Motivation

- Given person has disease, what is prob that he smoked?
- P(B|A)=P(BA)/P(A)=5/11
- Given person doesn't smoke, what is probability of disorder?
- P(A|notB)

  =P(A notB)/P(notB)

  =6/75

	Smoker B	Non-smoker $\overline{B}$	
Dis-order present A	5	6	11
Dis-order absent	20	69	89
	25	75	

Does smoking increase risk of disease?

- Given person has disease, what is prob that he smoked?

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$= .05/.25 = .2$$

- Given person doesn't smoke, what is probability of disorder?

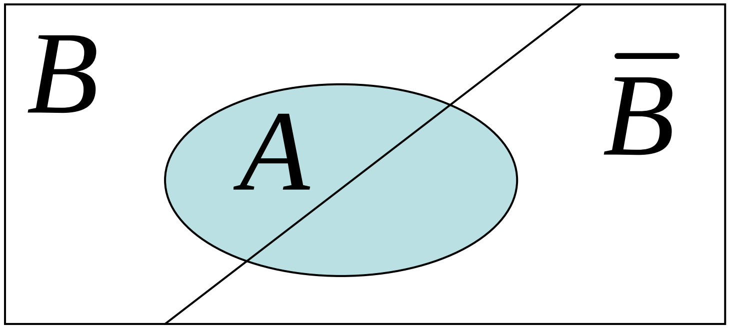
$$P(A|\bar{B}) = \frac{P(A\bar{B})}{P(\bar{B})}$$

$$= .06/.75 = .08$$

$\bar{B}$

	Smoker B	Non-smoker	
Dis-order present A	5	6	11
Dis-order absent	20	29	89
	25	69	

Tricks with Conditional Prob.



$$P(A) = P(AB) + P(A\bar{B})$$

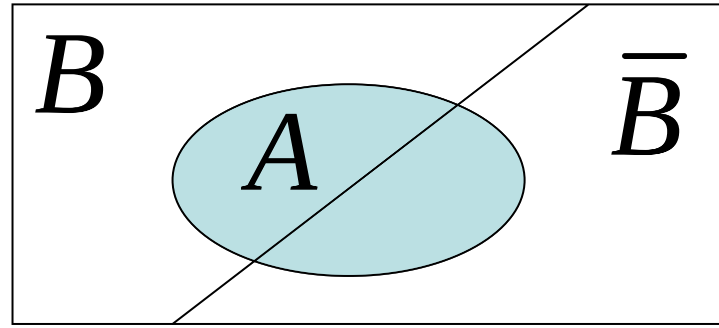
$$P(A \mid B) = \frac{P(AB)}{P(B)} \qquad P(B \mid A) = \frac{P(AB)}{P(A)}$$

$$P(AB) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

$$P(A\bar{B}) = P(A \mid \bar{B})\ \ P(\bar{B})$$

$$P(A) = P(A \mid B)P(B) + P(A \mid \bar{B})P(\bar{B})$$

Bayes Rule – 2 Events.



$$P(B|A) = \frac{P(A|B)\ P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}$$

Imperfect Clinical Trials

A disease is present in 1% of the population. A diagnostic test has success rate of 95% in detecting disease. Also its success rate is 95% in showing negative case.

If you test positive, what is the probability that you have the disease?

Set up Model

D=Have Disease

T=Test Positive

Want P(D|T)

Know

$$P(D) = .01 \qquad P(\bar{D}) = .99 \quad \text{since } P(D)+P(\bar{D})=1$$

$$P(T \mid D) = .95 \qquad P(\bar{T} \mid \bar{D}) = .95$$

$$P(\bar{T} \mid D) = .05 \qquad P(T \mid \bar{D}) = .05$$

Confusion Matrix

$$P(TD) = P(T \mid D)P(D)$$
$$= .95 * .01 = .0095$$

$$P(\overline{T}\overline{D}) = P(\overline{T} \mid \overline{D})P(\overline{D})$$
$$= .95 \cdot .99 = .9405$$

	D	$\overline{D}$	
T	.0095	.	
$\overline{T}$		.9405	.
.	.01	.99	1.0

Confusion Matrix

Posterior Probability

$$P(D|T) = \frac{P(DT)}{P(T)} = .16$$

Prior Probability

$$P(D) = .01$$

	D	$\bar{D}$	
T	.0095	.0495	.059
$\bar{T}$	.0005	.9405	.941
.	.01	.99	1.0

Compute directly:

$$P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|\bar{D})P(\bar{D})} = .16$$

Confusion Matrix

What if you test negatively?

	D	$\bar{D}$	
T	.0095	.0495	.059
$\bar{T}$	.0005	.9405	.941
.	.01	.99	1.0

$$P(D|\bar{T}) = \frac{P(T\bar{D})}{P(\bar{T})} = \frac{.0005}{.941} = .00053$$

Bayes Rule Applied to Last Example

- A= Has disease
- B= Smokes

$$P(B \mid A) = .95 \quad P(\bar{B} \mid \bar{A}) = .95 \quad P(A) = .01$$

$$P(A \mid B.) = \frac{P(B \mid A)\ P(A)}{P(A \mid B)P(B) + P(A \mid \bar{B})P(\bar{B})}$$

$$= \frac{.95 * .01}{.01 * .95 + .99 * .05} = .16$$

General Bayes Rule

- Let $A_1, A_2, \ldots, A_n$ be disjoint events that form a partition of the sample space

$$P(A_i \mid B) = \frac{P(B \mid A_i) P(A_i)}{\sum_j P(B \mid A_j) P(A_j)}$$

Example

Three production lines for cell phones produce 50%, 30%, and 20% of total output respectively.

Each line produces 5%, 15%, 10% of products defective respectively.

A cell phone is selected at random and found to be defective. What is the probability it comes from line 1?

Answer

Events

L1, L2, L3 = made by line I

D = Defective

P(L1)=.5 P(L2)=.3 P(L3)=.2

P(D|L1)=.05 P(D|L2)=.15 P(D|L3)=.1

$$P(L1 \mid D) = \frac{P(D \mid L1)P(L1)}{\sum_i P(D \mid Li)P(Li)}$$

$$= \frac{.05 * .5}{.05 * .5 + .15 * .3 + .1 * .2} = .28$$

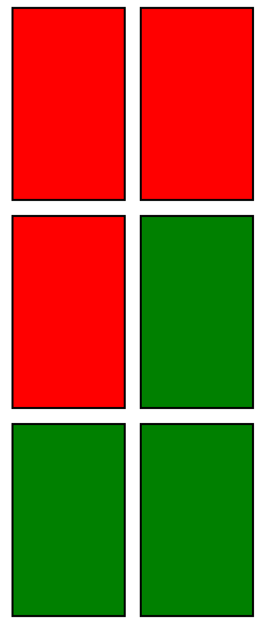
In Class Exercise

Given 3 cards: one has red on both sides; one has green on both sides and one has red on one side and green on other side. Pick a card. Observe one side.If the first side observed is green, what is the probability that second side observed is green.

Solve by guessing, experimentation, and Bayes rule.

Experiment

- Card 1
- Card 2
- Card 3



Pick a card randomly. Observe one side randomly. If you observe first side is green, what is the probability that the second side is green?