Computational Optimization

Constrained Optimization

Easiest Problem

- Linear equality constraints

$$\min f(x) \qquad f \in R^n$$

$$s.t. \quad Ax = b \qquad A \in R^{m\times n},\ b \in R^m$$

Null Space Representation

- Let x* be a feasible point, $Ax^*=b$.
- Any other feasible point can be written as $x=x^*+P$ where $Ap=0$
- The feasible region

  $\{x : x^*+p \quad p \in N(A)\}$

  where N(A) is null space of A

Example

$$\min \tfrac{1}{2}\left(x_1^2 + x_2^2 + x_2^3\right)$$

$$s.t.\quad x_1 + 3x_2 + 4x_3 = 4$$

- Solve by substitution

$$\min \tfrac{1}{2}\left(x_1^2 + x_2^2 + x_2^3\right)$$

$$s.t.\quad x_1 = 4 - 3x_2 - 4x_3$$

becomes

$$\min \tfrac{1}{2}\left(\left(4 - 3x_2 - 4x_3\right)_3^2 + x_2^2 + x_2^3\right)$$

Null Space Method

$$\min \tfrac{1}{2}\left(x_1^2 + x_2^2 + x_2^3\right)$$

$$s.t.\quad x_1 + 3x_2 + 4x_3 = 4$$

- x*= [4 0 0]'

$$Z = \begin{bmatrix} -3 & -4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- x=x*+v

$$\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 & -4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 4 - 3v_1 - 4v_1 \\ v_1 \\ v_2 \end{bmatrix}$$

becomes

$$\min \tfrac{1}{2}\left(\left(4 - 3v_1 - 4v_2\right)_3^2 + v_1^2 + v_3^3\right)$$

Variable Reduction Method

- Let A=[B N] for x* (a basic feasible solution with at most m nonzero variables corresponding to columns of B) $A \in R^{m\times n}\, B \in R^{m\times m}\;\; N \in R^{m\times(n-m)}\;\; I \in R^{(n-m)\times(n-m)}$

$$Z = \begin{bmatrix} -B^{-1}N \\ I \end{bmatrix}$$

assumes m < n

is a basis matrix for null space of A

$$A_r = \begin{bmatrix} B^{-1} \\ 0 \end{bmatrix} \quad \Rightarrow AA_r = \begin{bmatrix} B\, N \end{bmatrix}\begin{bmatrix} -B^{-1} \\ 0 \end{bmatrix} = I + 0$$

Where did Z come from?

A=[1 3 4]   x* = [4 0 0]

A=[B N]

B=[1] N = [3 4]

$$Z = \begin{bmatrix} -B^{-1}N \\ I \end{bmatrix} = \begin{bmatrix} -1*[3\ 4] \\ 1 \quad 0 \\ 0 \quad 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

General Method

- There exists a Null Space Matrix

$$Z \in R^{n \times r} \qquad r \geq n - m$$

- The feasible region is:

$$\{x \mid x^* + Zv\}$$

- Equivalent "Reduced" Problem

$$\min_v f(x^* + Zv)$$

Practice Problem

$$\min \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$$

$$s.t. \quad x_1 + 3x_2 + 4x_3 = 4$$

$$x_2 - x_3 = 0$$

Optimality Conditions

- Assume feasible point and convert to null space formulation

$$g(v) = f(x^* + Zv)$$

$$\nabla g(v) = Z'\nabla f(x^* + Zv) = Z'\nabla f(y) = 0 \qquad where\, z = x^* + Zv$$

$$\nabla^2 g(v) = Z'\nabla^2 f(x^* + Zv)Z = Z'\nabla^2 f(y)Z$$

Lemma 14.1 Necessary Conditions (Nash + Sofer)

- If x* is a local min of f over {x|Ax=b}, and Z is a null matrix

$$\Rightarrow Z'\nabla f(x^*) = 0$$
$$and \;\; Z'\nabla^2 f(x^*)Z \; is \;\; p.s.d.$$

- Or equivalently use KKT Conditions

$$\Rightarrow \left.\begin{array}{l} \nabla f(x^*) - A'\lambda = 0 \\ Ax^* = b \end{array}\right\rangle \;\; has\, a\, solution$$
$$\Rightarrow Z'\nabla f^2(x^*)Z \;\; is \;\; p.s.d.$$

Lemma 14.2 Sufficient Conditions (Nash + Sofer)

- If x* satisfies (where Z is a basis matrix for Null(A))

$$Ax^* = b$$

$$Z'\nabla f(x^*) = 0$$

$$Z'\nabla^2 f(x^*)Z \quad is \;\; p.d.$$

then x* is a strict local minimizer

Lemma 14.2 Sufficient Conditions (KKT form)

- If (x*,λ*) satisfies (where Z is a basis matrix for Null(A))

$$Ax^* = b$$

$$\nabla f(x^*) - A'\lambda = 0$$

$$Z'\nabla^2 f(x^*)Z \quad is \;\; p.d.$$

then x* is a strict local minimizer

Lagrangian Multiplier

- λ* is called the Lagrangian Multiplier
- It represents the sensitivity of solution to small perturbations of constraints

$$f(\hat{x}) \approx f(x^*) + (\hat{x} - x^*)' \nabla f(x^*)$$

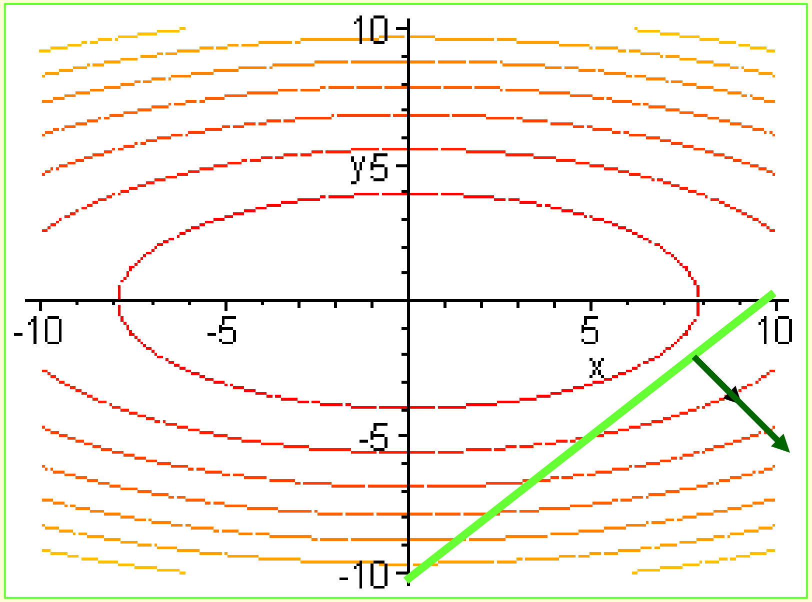
$$= f(x^*) + (\hat{x} - x^*)' A' \lambda^* \quad by \ KKT \ OC$$

$$Now \ let \ A\hat{x} = b + \delta$$

$$= f(x^*) + \delta' \lambda^* = f(x^*) + \sum_{i=1}^{m} \delta_i \lambda_i^*$$

Optimality conditions

- Consider min $(x^2+4y^2)/2$ s.t. $x-y=10$



$$\nabla f(x) - A'\lambda = 0$$

$$Ax = b$$

$$\Updownarrow$$

$$\begin{bmatrix} x \\ 4y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \lambda$$

$$x - y = 10$$

$$\Updownarrow$$

$$x^* = \lambda^* = 8,\ y^* = 2,$$

Optimality conditions

- Find KKT point

$$\nabla f(x) - A'\lambda = 0$$
$$Ax = b$$
$$\Updownarrow$$
$$\begin{bmatrix} x \\ 4y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \lambda$$
$$x - y = 10$$
$$\Updownarrow$$
$$x^* = \lambda^* = 8, y^* = -2,$$

Check SOSC

$$Z' = [1 \ 1]$$
$$\nabla^2 f(x) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$
$$Z'\nabla^2 f(x) Z \ \ is \ \ p.d.$$

So SOSC satisfied

Or we could just observe that it is a convex program so FONC are sufficient

Linear Equality Constraints - I

$$\min \frac{1}{2}\left(x_1^{\ 2} + 4x_2^{\ 2}\right)$$

$$\text{s.t. } x_1 - x_2 = 10$$

$$\nabla f(x) = A^T \lambda \qquad \mathrm{Ax = b}$$

$$\nabla \mathrm{f(x)} = \begin{bmatrix} \mathrm{x}_1 \\ 4\mathrm{x}_2 \end{bmatrix} \qquad \mathrm{A} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$\therefore$$

$$\begin{bmatrix} x_1 \\ 4x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \lambda$$

$$x_1 - x_2 = 10$$

Linear Equality Constraints - II

Solve:

$$x_1 = \lambda \Rightarrow 4x_2 = -x_1 \Rightarrow x_1 = -4x_2$$

$$-4x_2 - x_2 = 10 \Rightarrow$$

$$-5x_2 = 10$$

$$x_1 = 8$$

$$x_2 = -2$$

$$\lambda = 8$$

$$x^* = \begin{bmatrix} 8 \\ -2 \end{bmatrix}, \quad \lambda^* = 8 \quad \leftarrow \text{KKT point}$$

Linear Equality Constraints - III

SOSC $\quad A = \begin{bmatrix} 1 & -1 \end{bmatrix} \quad Z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\nabla^2 f(x) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$Z^T \nabla^2 f(x) Z = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} > 0$$

so SOSC satisfied, and x* is a strict local minimum

Objective is convex, so KKT conditions are sufficient.

Handy ways to compute Null Space

- Variable Reduction Method
- Orthogonal Projection Matrix
- QR factorization (best numerically)
- Z=Null(A) in matlab

Orthogonal Projection Method

- Use optimization. Minimize distance between given point c and null space of A.

$$\min_p \quad \frac{1}{2}\left\| p - c \right\|^2$$
$$s.t. \qquad Ap = 0$$

$$\nabla f(p^*) = A'\lambda$$
$$Ap^* = 0$$
or equivalently
$$(p^* - c) = A'\lambda$$
$$Ap^* = 0$$

Orthogonal Projection Method

Optimality conditions give us the solution

$$FONC\ is$$

$$(p^* - c) = A'\lambda$$

$$Ap^* = 0$$

$$\Rightarrow Ap^* - Ac = AA'\lambda$$

$$\Rightarrow \lambda = -\left(AA'\right)^{-1} Ac$$

$$\Rightarrow p^* = A'\lambda + c = -A'\left(AA'\right)^{-1} Ac + c$$

$$= (I - A'\left(AA'\right)^{-1} A)c$$

Orthogonal Projection Method

Final result is:

$$(I - A'\left(AA'\right)^{-1} A) \in Null\ \ Matrices\ of\ \ A$$

Note null space matrix is not unique

Try it in Matlab for A= [1 3 5; 2 4 -1]
Compare with Null(A) Null('A',r)

Get Lagrangian Multipliers for free!

- The matrix

$$A_r = A'(AA')^{-1} \quad where \ AA_r = AA'(AA')^{-1} = I$$

is the right inverse matrix for A.

- For general problems

$$\min f(x) \ \ s.t. \ Ax = b$$

$$\lambda^* = A_r' \nabla f(x^*)$$

Let's try it

- For

$$\min\ f(x) = \tfrac{1}{2}\left[x_1^2 + x_2^2 + x_3^2 + x_4^2\right]$$

$$s.t. \qquad x_1 + x_2 + x_3 + x_4 = 1$$

- Projection matrix

$$Z = I - A'\left(AA'\right)^{-1}A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\left[\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]^{-1}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

Solve FONC for Optimal Point

- FONC

$$\nabla f(x) - A'\lambda = \begin{bmatrix} x_1 \\ x_2 \\ x \\ x_4 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_1 + x_2 + x_3 + x_4 = 1$$

Check Optimality Conditions

- For

$$x^* = [1111]/4$$

$$\nabla f(x^*) = [1111]/4$$

$$Z\nabla f(x^*) = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} * \frac{1}{4} = 0$$

$$Ax^* = b$$

Using Lagrangian

$$A_r = A'(AA')^{-1} = [1111]'\frac{1}{4}$$

$$\lambda = A_r \nabla f(x^*) = 1/4$$

$$Clearly$$

$$\nabla f(x^*) = A'\lambda$$

You try it

$$\min \; f(x) = \tfrac{1}{2} x' C x$$
$$s.t. \qquad Ax = b$$

- For

$$C = \begin{bmatrix} 0 & -13 & -6 & -3 \\ -13 & 23 & -9 & 3 \\ -6 & -9 & -12 & 1 \\ -3 & 3 & 1 & 3 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1 & 3 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

- Find projection matrix
- Confirm optimality conds are Z'Cx*=0,

  Ax* = b
- Find x*
- Compute Lagrangian multipliers

  Check Lagrangian form of the multipliers.

Variable Reduction Method

- Let A=[B N]   A is m by n B is m by m
  assume m < n

$$Z = \begin{bmatrix} -B^{-1}N \\ I \end{bmatrix}$$

is a basis matrix for null space of A

$$A_r = \begin{bmatrix} B^{-1} \\ 0 \end{bmatrix} \quad \Rightarrow AA_r = [B\,N]\begin{bmatrix} B^{-1} \\ 0 \end{bmatrix} = I + 0$$

Try on our example

- Take for example first two columns for B

$$A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} B & N \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \end{bmatrix}$$

- Then

$$Z = \begin{bmatrix} 1 & -2 \\ -4 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A_r = \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Condition number of Z' CZ = 158

  better but not great

QR Factorization

- Use Gram-Schmidt algorithm to make orthogonal factorize A'=QR with Q orthogonal and R upper triangular

$$A' = QR = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$$

$$\text{where } A \in m \times n, Q_1 \in n \times m, Q_2 \in n \times (n-m), R_1 \in m \times m$$

$$Z = Q_2 \qquad A_r = Q_1 R_1^{-T}$$

QR on problem

- Use matlab command QR

  [Q R] = qr(A')

  Q2 = Q(:,3:4)

Cond(Q2'*C*Q2) = 9.79

In Class Practice

- Find optimal solution and verify FONC and SOSC of the following:
- Let the perimeter of a rectangle be fixed to 4. Find the shape of the rectangle with largest area.
- Solve the problem

$$\max \quad x_1x_2+x_2x_3+x_1x_3$$

$$\text{s.t.} \quad x_1+x_2+x_3=3$$