Computational Optimization

Laboratory 2 - 1/29/08- Quadratic Function and Optimality Conditions

Due in class Tuesday 2/5/08

I will start a sentence with three asterisks if I want you to turn in an answer to the question posed. Otherwise, you should be able to answer the question for your own edification. For the matlab exercise with three asterisks, please make a diary of your run and turn in any matlab .m files that you right. Feel free to edit and concatenate diary files as necessary for succinct presentation.

1. To Matlab all data is a matrix. So let's start be entering three 2 by 2 matrices and 2 column vectors.

```
>>  Q1 = [ 2 1; 1 3]
>>  Q2 = [-5 2 ; 2 -4]
>>  Q3 = [4 1; 1 -3]
>>  b = [1 3]'
>>  s = [ 2 1]'
```

   Note that ' denotes transpose. All the basic arithmetic operators are well defined. You can add and multiply matrices as long as their sizes are appropriate. Try:

```
>>  Q1+Q2
>>  2*b
>>  Q1*b
>>  b*Q1
```

   Why does the last expression give you an error?

2. Let's say we wanted to minimize functions of the form $1/2x'Qx - b'x$. The file qp.m contains the following code:

```
%This is a function to calculate 1/2 x'Qx-b'x
function  qpx = qp(Q,b,x)
qpx = 0.5*x'*Q*x -b'*x;
```

   Try evaluating the qp function $1/2x'Q1x - b'x$

```
>> qp(Q1,b,s)
```

   Now try it on the function $1/2x'Q2x - b'x$.

```
>> qp(Q2,b,s)
```

3. We know a necessary condition for a point, X, to be a minimum or a maximum of such a quadratic function is that $\nabla f(x) = Qx - b = 0$. We can use Matlab to compute such a point for $Q1$. The following command has Matlab find the inverse matrix of Q1 and multiply it by b to solve for b.

```
>> y1 = inv(Q1)*b
```

The exact solution to the problem is [0,1]'. You can check that the value for $y1$ is accurate by typing

```
>> Q1*y1-b
```

Notice that Matlab multiplied the entire vector by a scalar when giving the solution. The last command should give you something close to the zero vector. An alternate way to solve for $y1$ is:

```
>> y1 = Q1\b
```

This command solves for $y1$ using Gaussian elimination on the linear system $Q1y1 = b$.

4. Since this is a two dimensional problem we can plot the results. This sequence of commands will solve for y1 and then plot the result on a surface plot of the function with a contour plot shown as well. Note that you have to create a mesh of points X,Y and then define Z on this mesh before you can do a 3D plot or contour, and you might want to rotate the plot.

```
y1 = Q1\b;
fy1=1/2*y1'*Q1*y1-b'*y1;

[X Y]=meshgrid(-10:.2:10,-10:.2:10);
Z=1/2*(Q1(1,1)*X.^2+2*Q1(1,2)*X.*Y+Q1(2,2)*Y.^2)-b(1)*X-b(2)*Y;%

surfc(X,Y,Z)
hold
plot3(y1(1),y1(2),fy1,'r*')
hold
```

5. The second order sufficient conditions allow us to check if a stationary point is a strict local minimum. We can do this by analyzing the Hessian matrix. We know the following chart holds true.

All eigenvalues	Hessian
≥ 0	positive semi-definite
> 0	positive definite
≤ 0	negative semi-definite
< 0	negative definite
o.w.	indefinite

Note that for a quadratic function in the form $0.5 * x'Q * x - b' * x$ that the Hessian matrix is just Q. To check the eigenvalues of Q1 type

```
>> eig(Q1)
```

**Is $y1$ a strict local minimum or maximum? What is the greatest claim can you make about $y1$ in terms of it being a strict, local, and/or global minimum/maximum? Prove your claim. Hint: a graph is not a proof and you must use second order information.

6. **Repeat the Problems 4 and 5 for the functions $0.5*x'Q2*x-b'*x$ and $0.5*x'Q3*x-b'*x$.

7. ** **For graduate credit only.** Consider the following problem: Let $g_1, g_2, \ldots, g_m$ be concave functions on R^n. Let f be a convex function on R^n, and μ a positive constant. Prove that function

$$\beta(x) = f(x) - \mu \sum_{i=1}^{m} log g_i(x) \tag{1}$$

is convex on the set $S = \{x : g_i(x) > 0,\ i = 1, \ldots, m\}$.

8. ** Consider the function $f(x_1, x_2) = 2x_1^2 + x_2^2 - 2*x_1*x_2 + 2x_1^3 + x_1^4$. Find all of the stationary points of this function and indicate what kind of minima or maxima they are (local, global, strict, none of the above, etc). Determine whether the first order and second order optimality conditions are satisfied at each stationary point.